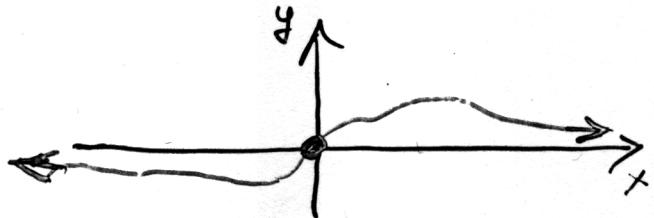


Pl. Näherlich ist $f(x) = x e^{-\frac{x^2}{2}}$ gut!

I. $D_f = \mathbb{R}$, f parallen, $x=0$ ggöL. Es gilt $x \geq 0 \rightarrow$ mit gelten.
 $\lim_{x \rightarrow \infty} x e^{-\frac{x^2}{2}} = \lim_{t \rightarrow \infty} \sqrt{2t} e^{-t} = \lim_{t \rightarrow \infty} \frac{\sqrt{2t}}{e^t} = \lim_{t \rightarrow \infty} \frac{\frac{1}{2}\sqrt{2t}}{e^t} = 0$

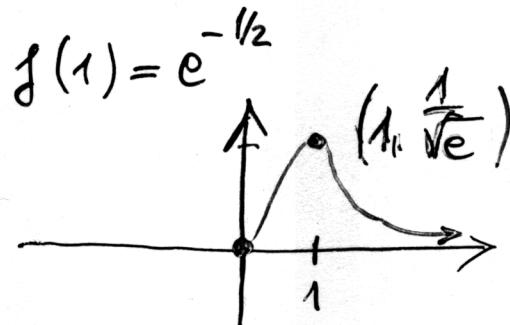


II. $D_{f'} = \mathbb{R}$.

$$f'(x) = e^{-\frac{x^2}{2}} + x e^{-\frac{x^2}{2}} (-\frac{1}{2} \cdot 2x) = (1-x^2) e^{-\frac{x^2}{2}}$$

$$f'(x) = 0 \Rightarrow x = 1 \geq 0, x = -1 < 0$$

x	$(0, 1)$	1	$(1, \infty)$
$f'(x)$	+	-	
$f(x)$	\nearrow	\searrow	lok. maximum.



III. $D_{f''} = \mathbb{R}$

$$f''(x) = -2x e^{-\frac{x^2}{2}} + (1-x^2) e^{-\frac{x^2}{2}} (-x) =$$

$$= e^{-\frac{x^2}{2}} (2+1-x^2)(-x) = \checkmark$$

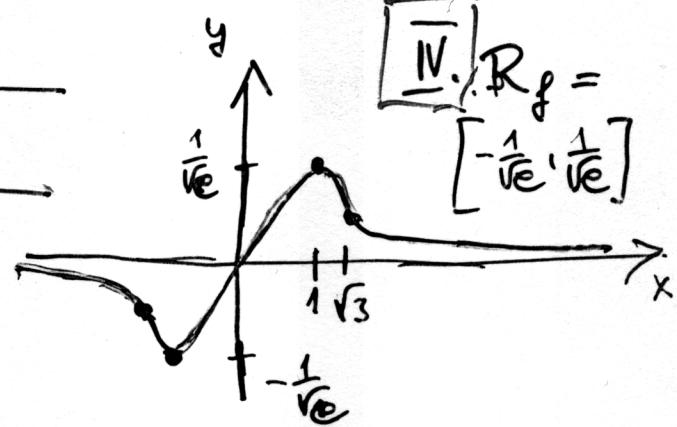
$$\stackrel{!}{=} e^{-\frac{x^2}{2}} (x^3 - 3x) = e^{-\frac{x^2}{2}} (-x)(x^2 - 3).$$

$$f''(x) = 0, \text{ bei } x=0 \text{ bzw. } x=\sqrt{3} > 0.$$

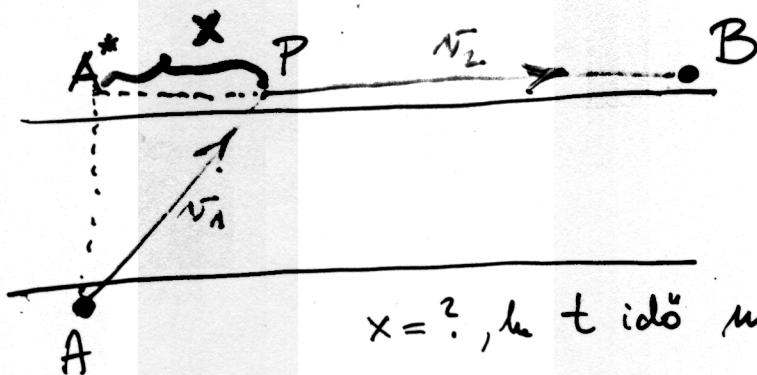
x	0	$(0, \sqrt{3})$	$\sqrt{3}$	$(\sqrt{3}, \infty)$
$f''(x)$	0	-	0	+
$f'(x)$	\searrow		\nearrow	
$f(x)$	0	inflexion	inflexion	

$$f(\sqrt{3}) = \sqrt{3} e^{-\frac{3}{2}}$$

IV. $R_f = [-\frac{1}{\sqrt{e}}, \frac{1}{\sqrt{e}}]$



Abszolut bővöntér



1. Példa

$$AB = 10 \text{ km}$$

$$AA' = 4 \text{ km}$$

$$\therefore v_1 = 2 \text{ km/ó}$$

$$v_2 = 6 \text{ km/ó}$$

$$\overline{AP} = \sqrt{16+x^2} \quad \overline{BP} = 10-x$$

$$t = \frac{\sqrt{16+x^2}}{v_1} + \frac{10-x}{v_2} = \underbrace{\frac{\sqrt{16+x^2}}{2}}_{\text{folyt.}} + \frac{10-x}{6} = t(x) \quad x \in [0, 10] = I.$$

\Rightarrow van minimuma.

$$t'(x) = \frac{1}{2\sqrt{16+x^2}} \cdot 2x - \frac{1}{6} = 0$$

$$3x = \sqrt{16+x^2}$$

$$9x^2 = 16+x^2$$

$$8x^2 = 16 \quad x^2 = 2 \quad x = \pm \sqrt{2}$$

$$x = \sqrt{2} \in I$$

$$t(0) = 2 + \frac{5}{3} = \frac{11}{3} = 3.6666$$

$$t(\sqrt{2}) = \frac{\sqrt{18}}{2} + \frac{10-\sqrt{2}}{6} = \frac{5}{3} + \frac{3\sqrt{2}}{2} - \frac{\sqrt{2}}{6} = \underline{\underline{\frac{5}{3} + \frac{4}{3}\sqrt{2}}} \approx 3.5522$$

$$t(10) = \frac{\sqrt{116}}{2} = \sqrt{29} \approx 5.3851 \quad \text{minimális.}$$

2. Példa Adott gömbbe írható szabályos kúpok közt melyiknek a legnagyobb a térfogata?



$$V = \frac{\pi r^2 \cdot m}{3} \quad (m-R)^2 + r^2 = R^2 \text{ adott.}$$

$$r^2 = R^2 - (m-R)^2$$

$$V = (R^2 - (m-R)^2) \pi \cdot \frac{m}{3} = V(m), \quad m \in (0, 2R)$$

$$V = V(m) = \frac{\pi}{3} (2m^2 R - m^3) \max m = ?$$

$$V' = \frac{\pi}{3} (4mR - 3m^2) = 0 \quad m=0 \text{ nem m. o.} \quad m = \frac{4R}{3}$$

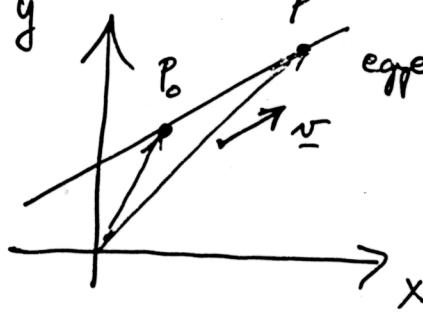
$$V(0) = 0 \quad V(2R) = 0 \quad V\left(\frac{4R}{3}\right) = \frac{\pi}{3} \frac{16R^2}{9} \left(2R - \frac{4R}{3}\right) = \frac{32}{81} \pi R^3$$

\Rightarrow szerint van maximuma.

Ez csak a $V\left(\frac{4R}{3}\right)$ lehet.

maximális térfogat.

Görbely megadása parameteresek:

① 

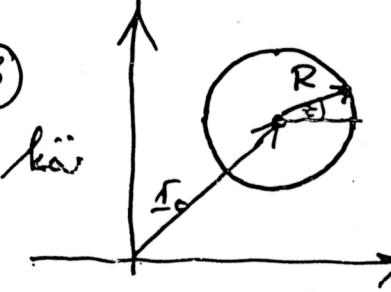
egyszer $P(\underline{t})$, $P(\underline{t}_0)$, $\overline{PP_0} \parallel \underline{v}$

$$\underline{t} - \underline{t}_0 = t \underline{v}$$

$$\underline{t} = \underline{t}_0 + t \underline{v}$$

$$\begin{cases} x = x_0 + t v_1 \\ y = y_0 + t v_2 \end{cases}, t \in \mathbb{R}.$$

② $y = f(x)$, $x \in (a, b) \Rightarrow x = t$, $y = f(t)$
 egyszerűbb grafikai.

③ 

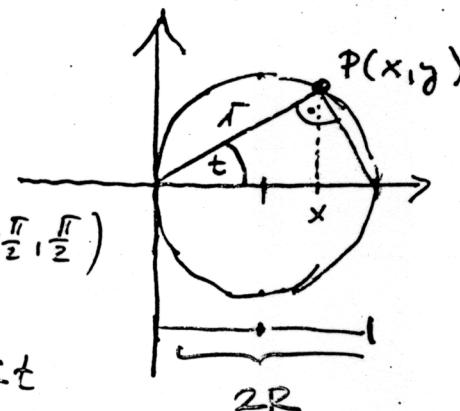
$$\underline{t} = \underline{t}_0 + R (\cos \underline{i} + \sin \underline{j})$$

$$x = x_0 + R \cos t$$

$$y = y_0 + R \sin t, t \in [0, 2\pi]$$

④ Speciális hörök.

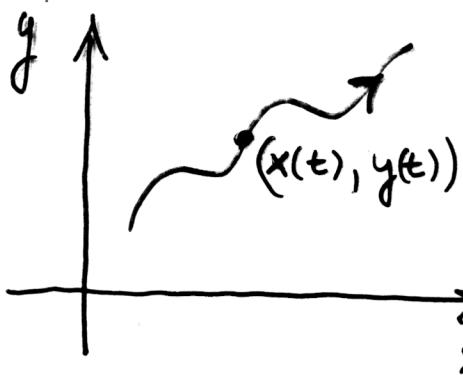
$$\begin{aligned} r &= 2R \cos t, t \in \left[\frac{\pi}{2}, \frac{\pi}{2} \right) \\ x &= r \cos t = 2R \cos^2 t \\ y &= r \sin t = 2R \cos t \sin t \end{aligned}$$



$$\begin{aligned} x &= R(1 + \cos 2t) \\ y &= R \sin 2t \\ -\frac{\pi}{2} &\leq t < \frac{\pi}{2} \end{aligned}$$

⑤ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ellipszis $\begin{aligned} x &= a \cos t, \\ y &= b \sin t, \end{aligned} t \in [0, 2\pi)$

⑥ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ hipérbolák $\begin{aligned} x &= a \cosh t, \\ y &= b \sinh t, \end{aligned} t \in \mathbb{R}$



Görbék paraméteres megadása.

$$x = x(t) = \varphi(t) ,$$

$$y = y(t) = \psi(t) , \quad t \in I = [t_1, t_2]$$

Feltevésük, hogy $\dot{\varphi}(t) > 0$, $t \in I$,
és $\exists \dot{\psi}(t)$, $t \in I$.

($\dot{\varphi} > 0 \Rightarrow x$ szig. mon. nö')

$$\Rightarrow \varphi - \text{műr } \exists \text{ inverze: } \varphi^{-1} \Rightarrow t = \varphi^{-1}(x) \Rightarrow$$

$$\Rightarrow y = \psi(t) \Big|_{t=\varphi^{-1}(x)} = \psi(\varphi^{-1}(x)) = (\psi \circ \varphi^{-1})(x) = f(x)$$

Tehát ez a görbészakasz megadásához $y = f(x)$ egyenlettelől
a grafikonjához is, $x \in [\varphi(t_1), \varphi(t_2)]$.

$$f'(x) \Big|_{x_0} = \frac{dy}{dx} \Big|_{x_0} = \dot{\psi}(t_0) (\varphi^{-1}(x))'_{x_0} =$$

$$= \dot{\psi}(t_0) \cdot \frac{1}{\dot{\varphi}(t_0)} = \frac{\dot{\psi}}{\dot{\varphi}} \Big|_{t_0} = \frac{\dot{y}}{\dot{x}} \Big|_{t_0}$$

ahol $\varphi(t_0) = x_0$ Tehát:

$$m = \boxed{\frac{dy}{dx} \Big|_{x_0} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \Big|_{t_0} \quad x(t_0) = x_0}$$

(x_0, y_0) -on
áthaladás
érintőegyen
meredeksége.

Megjegyzés: $\dot{x} \underline{=} + \dot{y} \underline{j}$ párhuzamos az érintőegyenettel.
érintővektor

Parametrisieren addt g rbe konvexit sinak visszalat.

$x = \varphi(t)$, $y = \psi(t)$, $t \in [t_1, t_2]$  t n \$\ddot{\varphi}, \ddot{\psi}\$ es folytonosak.
 $\dot{\varphi}(t) > 0$, $t \in [t_1, t_2]$.

$$y = \psi(t) \Big|_{t=\varphi^{-1}(x)} = \psi(\varphi^{-1}(x)) = f(x), \quad x \in [\varphi(t_1), \varphi(t_2)]$$

$$\begin{aligned} f''(x) \Big|_{x_0} &= \frac{d^2 y}{dx^2} \Big|_{x_0} = \frac{d}{dx} f'(x) \Big|_{x_0} = \frac{d}{dx} \frac{\dot{\psi}}{\dot{\varphi}} \Big|_{x_0} = \\ &= \frac{d}{dt} \frac{\dot{\psi}}{\dot{\varphi}} \cdot \frac{dt}{dx} \Big|_{x_0} = \frac{\ddot{\psi}\dot{\varphi} - \ddot{\varphi}\dot{\psi}}{\dot{\varphi}^2} \Big|_{t_0} \frac{1}{\dot{\varphi}|_{t_0}} \\ &= \frac{\ddot{\psi}\dot{\varphi} - \ddot{\varphi}\dot{\psi}}{\dot{\varphi}^3} \Big|_{t_0} \end{aligned}$$

$$\boxed{y'' = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^3}}$$

Hasonl an kapjuk juk:

$$\begin{aligned} y''' &= \frac{d}{dx} \left(\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^3} \right) = \frac{d}{dt} \left(\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^3} \right) \cdot \frac{dt}{dx} \\ &= \left(\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^3} \right)' \frac{1}{\dot{x}} = \text{stb.} \end{aligned}$$

Hc $y'' > 0 \Rightarrow$ a g rbe akkor l houwex

Hc $y'' < 0 \Rightarrow$ a g rbe akkor l konkav.

Pl. Vizsgáljuk meg az $x = t - \sin t$,
 $y = 1 - \cos t$, $t \in [0, 2\pi]$
 paraméteresben adott görbe eszközt!

$$\dot{x} = 1 - \cos t > 0, \quad (y = f(x) \text{ is lehető.})$$

$$y' = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\sin t}{1 - \cos t} = 0, \text{ ke } t = 0, \pi, 2\pi.$$

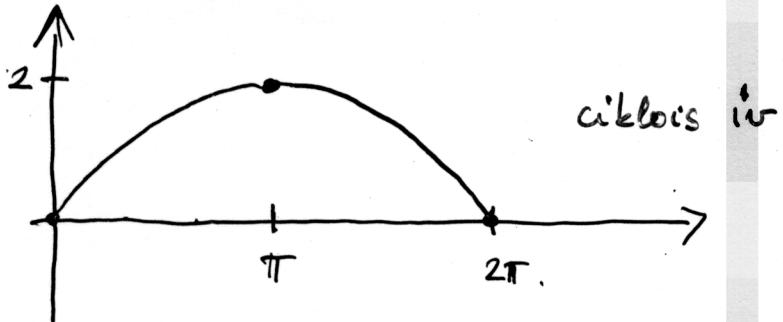
x	$(0, \pi)$	π	$(\pi, 2\pi)$
y'	+	0	-
y	↗ loc. max!	↓ loc. min!	↙

$$x = 0, \pi, 2\pi.$$

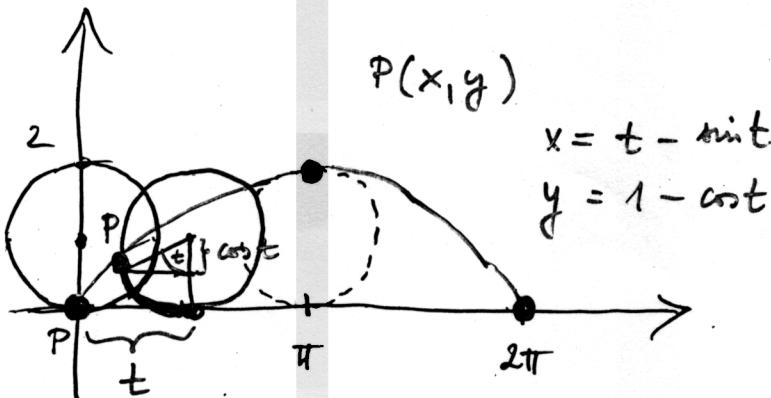
$$y = 0, 2, 0.$$

$$\begin{aligned}\dot{x} &= 1 - \cos t \\ \ddot{x} &= \sin t \\ \dot{y} &= \sin t \\ \ddot{y} &= \cos t\end{aligned}$$

$$\begin{aligned}y'' &= \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3} = \frac{\cos(1 - \cos t) - \sin \sin t}{(1 - \cos t)^3} = \frac{\cos - 1}{(1 - \cos t)} \\ &= \frac{-1}{(1 - \cos t)^2} < 0, \quad t \in (0, 2\pi) \Rightarrow f \text{ alulról körbev.}\end{aligned}$$



gyűrűi kerék
 mögött P pontjáról
 pályája.



(Pl)

$$\text{Legyen } x = \varphi(t) = t e^{2t},$$

$$y = \psi(t) = \frac{t+1}{2t+1}, \quad t > -\frac{1}{2}.$$

- a) Mutassuk meg, hogy a fenti, parameteres megoldású görbe megadható epp $y = f(x)$ formájában!
- b) Milyen lokális tulajdonságai vannak a f - mel az $x=0$ pontban?
- c) Van-e műffelxíójában f -mel az $x=0$ -ban?

$$\text{a) } \dot{\varphi}(t) = e^{2t} + 2t e^{2t} = (1+2t)e^{2t} > 0 \Rightarrow$$

$$\Rightarrow \varphi \text{ szig. mon. növő} \Rightarrow \exists \text{ az inverze } \varphi^{-1}$$

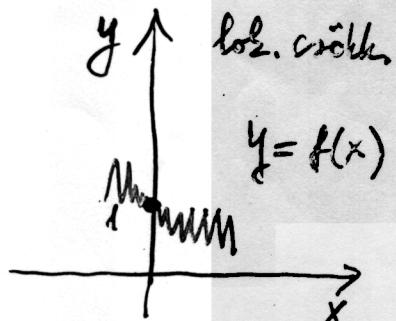
$$\Rightarrow t = \varphi^{-1}(x) \Rightarrow y = \psi(\varphi^{-1}(x)) = f(x)$$

$$f = \psi \circ \varphi^{-1}, \quad x \in (-\frac{1}{2}e^{-1}, \infty) = D_f$$

$$\text{b) } \left. \frac{dy}{dx} \right|_{x=0} = \left. \frac{\dot{y}}{\dot{x}} \right|_{t=0} = \left. \frac{\frac{2t+1-2(t+1)}{(2t+1)^2}}{(1+2t)e^{2t}} \right|_{t=0} = \left. \frac{-1}{e^0} \right|_{t=0} = -1 < 0$$

$\Rightarrow f$ lokálisan csökkenő az $x=0$ -ban.

$$\text{c) } \left. \frac{d^2y}{dx^2} \right|_0 = \left. \frac{\ddot{y} \dot{x} - \dot{y} \ddot{x}}{\dot{x}^3} \right|_{t=0} = ?$$



$$\dot{x} = (1+2t)e^{2t}$$

$$\ddot{x} = 2e^{2t} + (1+2t)2e^{2t}$$

$$\dot{y} = \frac{-1}{(2t+1)^2}$$

$$\ddot{y} = +2(2t+1)^{-3} \cdot 2$$

$$\dot{x}(0) = 1$$

$$\ddot{x}(0) = 2+2 = 4$$

$$\dot{y}(0) = -1$$

$$\ddot{y}(0) = 4$$

$$\left. \frac{d^2y}{dx^2} \right|_{x=0} = \left. \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^3} \right|_{t=0} = \left. \frac{4+4}{1^3} \right. = 8 \neq 0$$

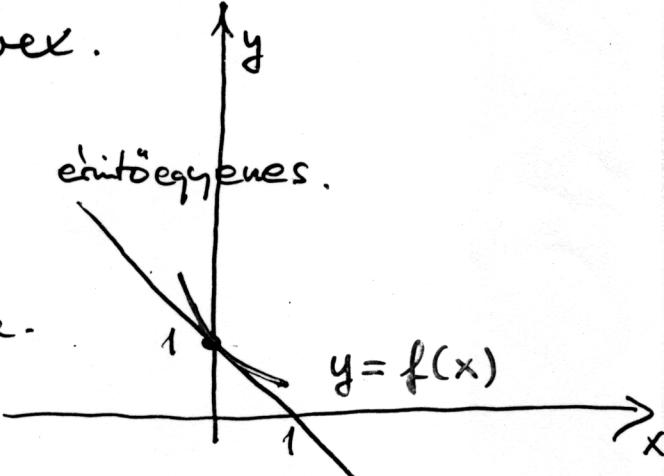
Az inflexió lefeszénet szükséges feltétele nem teljesül $\Rightarrow f$ -nek nincs inflexiója az $x=0$ -ban.

Megjegyzés: $x=0 \Rightarrow t=0 \Rightarrow y=1 \Rightarrow$
a görbe átmegy a $(0,1)$ ponton. Előzően említettem
már ezt a görbét: $m = \left. \frac{\dot{y}}{\dot{x}} \right|_0 = -1$

$\frac{d^2y}{dx^2} = f''(x)$ folytonos az $x=0$ hörnyezetében

és $f''(0) = 8 > 0 \Rightarrow f''(x)$ pozitív az $x=0$ egy
oldalán hörnyezetében \Rightarrow ebben a hörnyezetben
 f alakról konkav.

Hasonlóan leírható,
hogy f monoton csökkl.
az $x=0$ környéki hörnye-
zetében.



Láncszabály

$$(f \circ g)(x) = f(g(x)) = z$$

Ha g diffható x_0 -ban, f diffható $g(x_0)$ -ban

$\Rightarrow f \circ g$ is diff. ható x_0 -ban

$$\frac{d}{dx} |_{x_0} f \circ g = f' \Big|_{g(x_0)} \cdot g' \Big|_{x_0} =$$

$$= \frac{dy}{dx} \Big|_{g(x_0)} \cdot \frac{dz}{dy} \Big|_{x_0}$$

$$\boxed{\frac{d}{dx} |_{x_0} z = \frac{dz}{dy} \Big|_{y_0} \cdot \frac{dy}{dx} \Big|_{x_0}}$$

$$\left\{ \frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \right.$$

Inverz fv. deriválhatósági

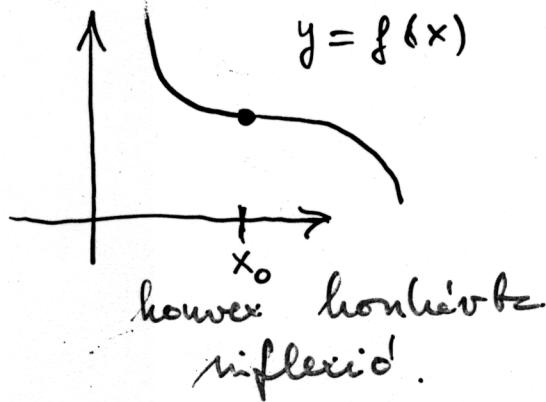
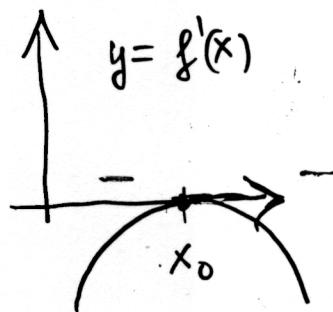
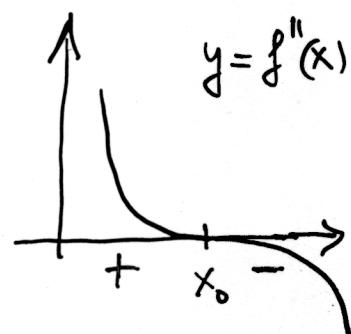
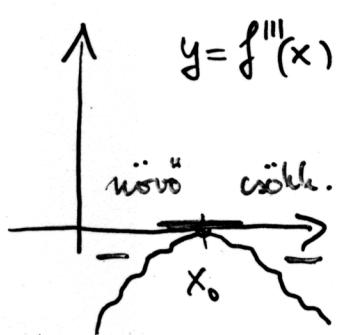
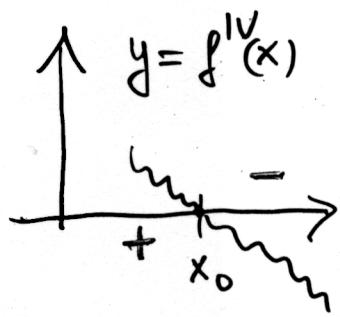
$$y = f(x) \quad x = f^{-1}(y)$$

$$f'(x) > 0$$

$$(f^{-1}(\))' \Big|_{y_0} = \frac{1}{f'()} \Big|_{x_0} \text{ ahol } y_0 = f(x_0)$$

$$\boxed{\frac{dx}{dy} \Big|_{y_0} = \frac{1}{\frac{dy}{dx} \Big|_{x_0}}} \quad \left\{ \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \right.$$

$$\text{Pl. } f'(x_0) = f''(x_0) = f'''(x_0) = f^{IV}(x_0) = 0 \quad f^V(x_0) < 0$$



Hf $f'(x_0) = f''(x_0) = f'''(x_0) = f^{IV}(x_0) = f^V(x_0) = 0$
 $f^{VI}(x_0) < 0$ ei $f(x_0) = -2$.

Milyen tulajdonságai f x_0 -ban?

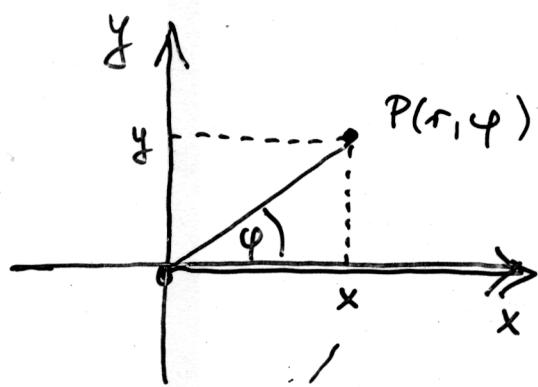
Hf legyen $f(x) = \begin{cases} 0, & \text{ha } x=0 \\ 2x^2 + x^2 \sin \frac{1}{x}, & \text{ha } x \neq 0 \end{cases}$

$$g(x) = \begin{cases} 0, & \text{ha } x=0 \\ 2x^3 + x^3 \sin \frac{1}{x}, & \text{ha } x \neq 0. \end{cases}$$

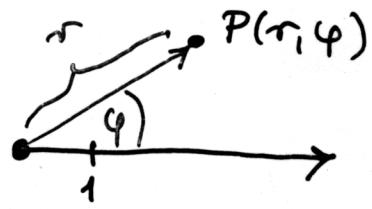
Milyen tulajdonsága van f -nek, illetve g -nek az $x_0=0$ -ban?

Hányanak differenciálható f az $x_0=0$ -ban?

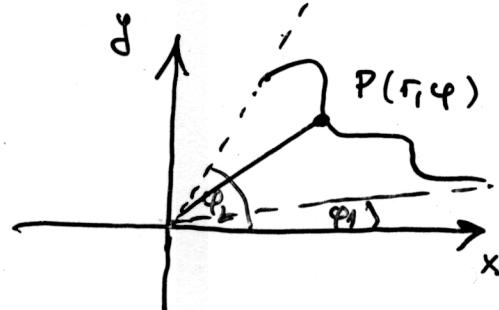
Görbék meg adásra polar koordináta rendszertben.



$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \end{cases}$$



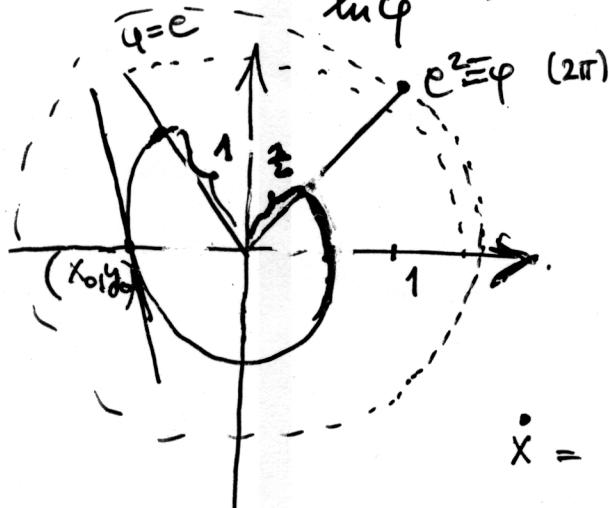
összefüggés a desz.
rögi Descartes koord.
és a polar koord.
között.



$$\boxed{\begin{aligned} \varphi_1 &\leq \varphi \leq \varphi_2 \\ r &= r(\varphi) \end{aligned}}$$

görte „polar“ alakja.

Pl. $r = \frac{1}{\ln \varphi}, \quad \varphi \in (c, c^2)$



$$\varphi_0 = \pi$$

$$x = \frac{1}{\ln \varphi} \cos \varphi$$

$$x_0 = \frac{1}{\ln \pi} \cos \pi = \frac{-1}{\ln 1}$$

$$y = \frac{1}{\ln \varphi} \sin \varphi$$

$$y_0 = \frac{1}{\ln \pi} \sin \pi = 0$$

$$\dot{x} = \frac{-1}{\ln^2 \varphi} \cdot \frac{1}{\varphi} \cos \varphi + \frac{1}{\ln \varphi} (-\sin \varphi)$$

$$\dot{y} = \frac{-1}{\ln^2 \varphi} \cdot \frac{1}{\varphi} \sin \varphi + \frac{1}{\ln \varphi} \cos \varphi$$

$$\dot{x}(\pi) = \frac{1}{\pi \ln^2 \pi}$$

$$m = \left. \frac{dy}{dx} \right|_{x_0} = \frac{\dot{y}(\pi)}{\dot{x}(\pi)} = -\pi \ln \pi$$

$$\dot{y}(\pi) = \frac{-1}{\ln \pi}$$

Erkönök egyenes egyenlete: $y = -\pi \ln \pi \left(x + \frac{1}{\ln \pi} \right)$ $y = -\pi x \ln \pi - \pi$

Pt. Welken kiekin delijdownga van as $r = \cos \varphi$
polar koordinatiban adott fr-nek a $\varphi_0 = \frac{\pi}{4}$ pontban?

$$x = r(\varphi) \cos \varphi = \cos^2 \varphi$$

$$x\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

$$y = r(\varphi) \sin \varphi = \cos \varphi \sin \varphi$$

$$y\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

$$\frac{dx}{d\varphi} = \dot{x} = -2 \cos \varphi \sin \varphi = -\sin 2\varphi \quad \frac{dy}{d\varphi} = \dot{y} = -\sin^2 \varphi + \cos^2 \varphi = \cos 2\varphi$$

$$\dot{x}\left(\frac{\pi}{4}\right) = -1 \quad \dot{y}\left(\frac{\pi}{4}\right) = 0$$

$$\frac{d^2x}{d\varphi^2} = \ddot{x} = -2 \cos 2\varphi$$

$$\frac{d^2y}{d\varphi^2} = \ddot{y} = -2 \sin 2\varphi$$

$$\ddot{x}\left(\frac{\pi}{4}\right) = 0$$

$$\ddot{y}\left(\frac{\pi}{4}\right) = -2$$

$$y' = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{0}{-1} = 0$$

$$y'' = \frac{d^2y}{dx^2} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{(\dot{x})^3} = \frac{2}{(-1)^3} = -2$$

(melsöttel gyanus legy)

$$y' = 0 \text{ és } y'' < 0 \Rightarrow$$

kiekin maximum.
nincs inflexioji.

$$y'' = \frac{-2\sin 2\varphi + 2 \cos 2\varphi}{-\sin^3 2\varphi}$$

folytonos a $\varphi = \frac{\pi}{4}$ körön-

lebenn.

Megjelezni: 1) A fr. alakzat konkr. az $x_0 = \frac{1}{2}$
hőringetetben.

Megjelezni:

$$2) x = \frac{\cos 2\varphi + 1}{2}$$

$$(2x-1)^2 + (2y)^2 = 1$$

$$y = \frac{\sin 2\varphi}{2}$$

$$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$$

körül van elo
az $(\frac{1}{2}, \frac{1}{2})$ pontban.

