$\boxed{-1-}$

1, $\boxed{13}$ $\displaystyle\sum_{k=1}^{\infty} 2k\, x^{2k-2} = \sum_{k=1}^{\infty} \underbrace{2k}_{a_{k-1}=2k}\, y^{k-1}$

$y = x^2$

$\sqrt[k+1]{|2k|} = \sqrt[k+1]{2} \cdot \sqrt[k+1]{k} \longrightarrow 1 \Rightarrow R_y = \frac{1}{1} = 1 \Rightarrow R_x = \sqrt{R_y} = \underline{\underline{1}}$ (2) (2)

$$\sum_{k=1}^{\infty} 2k\, x^{2k-2} = \frac{1}{x} \sum_{k=1}^{\infty} 2k\, x^{2k-1} \text{ (2)} = \frac{1}{x} \sum_{k=1}^{\infty} \frac{d}{dx}\left(x^{2k}\right) \text{ (2)} = \frac{1}{x}\frac{d}{dx}\left(\sum_{k=1}^{\infty} x^{2k}\right) \text{ (1)} =$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{x^2}{1-x^2}\right) \text{ (2)} = \frac{1}{x}\,\frac{2x(1-x^2)-x^2(-2x)}{(1-x^2)^2} = \underline{\underline{\frac{2}{(1-x^2)^2}}} \text{ (2)}$$

($x = 0$-ra is jó a képlet)

2, $\boxed{12}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{7^n}\frac{(4x)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \underbrace{\left(\frac{-4}{7}\right)^n \cdot \frac{1}{\sqrt{n}}}_{a_n} \cdot x^n$ (1)

$\sqrt[n]{|a_n|} = \frac{4}{7} \cdot \frac{1}{\sqrt[2n]{n}} \longrightarrow \frac{4}{7} \Rightarrow R = \frac{7}{4}$ (4)

$x = \frac{7}{4}$ esetén

$\displaystyle\sum_{n=1}^{\infty} \left(\frac{-4}{7}\right)^n \frac{1}{\sqrt{n}} \left(\frac{7}{4}\right)^n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ konvergens, mert Leibniz sor, de nem abszolút konv., mert $\sum_n \frac{1}{n^\alpha}$ konv. ha $\alpha > 1$, és itt $\alpha = \frac{1}{2}$. (2)

$x = -\frac{7}{4}$ esetén

$\displaystyle\sum_{n=1}^{\infty} \left(\frac{-4}{7}\right)^n \frac{1}{\sqrt{n}} \left(-\frac{7}{4}\right)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$ (2)

Konvergencia tartomány: $\left(-\frac{7}{4}, \frac{7}{4}\right]$ (1)

abszolút konv. tart.: $\left(-\frac{7}{4}, \frac{7}{4}\right)$ (2)

3, $\boxed{16}$

$$f(x)=\frac{1}{\sqrt{4+x^4}}=\left(4+x^4\right)^{-1/2}=\frac{1}{2}\left(1+\left(\frac{x}{\sqrt{2}}\right)^4\right)^{-1/2} \quad 1$$

a, $\boxed{10}$

$$f(x)=\frac{1}{2}\left(1+\frac{x^4}{4}\right)^{-1/2}=\frac{1}{2}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{x^4}{4}\right)^n=\underline{\underline{\sum_{n=0}^{\infty}\frac{1}{2}\binom{-1/2}{n}\frac{1}{4^n}x^{4n}}} \quad 3$$

Binomiális sor konv. sugara 1, 2 tehát $\left|\frac{x^4}{4}\right|<1$

$$\Rightarrow |x|<\sqrt[4]{4}=\underline{\underline{\sqrt{2}=R}} \quad 2$$

$4n=12 \Rightarrow n=3$

$$a_{12}=\frac{1}{2}\cdot\frac{1}{4^3}\binom{-1/2}{3}=\underline{\underline{\frac{1}{2^7}\,\frac{(-1/2)(-3/2)(-5/2)}{3\cdot 2}}} \quad 2$$

b, $\boxed{6}$

$$\int_0^1 f(x)\,dx=\int_0^1\left(\sum_{n=0}^{\infty}\frac{1}{2^{2n+1}}\binom{-1/2}{n}x^{4n}\right)dx=\sum_{n=0}^{\infty}\frac{1}{2^{2n+1}}\binom{-1/2}{n}\frac{1}{4n+1}$$

A kapott sor Leibniz-típusú. A sor első két tagjának összege adja

$$\int_0^1 T_4(x)\,dx=\frac{1}{2}\underbrace{\binom{-1/2}{0}}_{1}\frac{1}{1}+\frac{1}{8}\underbrace{\binom{-1/2}{1}}_{-\frac{1}{2}}\frac{1}{5}=\frac{1}{2}-\frac{1}{80}=\underline{\underline{\frac{39}{80}}} \quad 4$$

A hiba kisebb, mint a 3. tag abszolútértéke; $(n=2)$

$$H=\left|\int_0^1 f(x)\,dx-\int_0^1 T_4(x)\,dx\right|\le\underline{\underline{\frac{1}{2^5}\binom{-1/2}{2}\frac{1}{9}}} \quad 2$$

4, $\boxed{12}$

a, $\boxed{6}$ $f(x)=e^{-2x}=e^{-2(x-2)}\cdot e^{-4} \;2\; = e^{-4}\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}(x-2)^n \;3\;$; $R=\infty$ 1

$x_0=2$ $\qquad$ K.T. $=\mathbb{R}$

b, $\boxed{6}$ $g(x)=x^4e^{-2x}=x^4\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}x^n = \sum_{n=0}^{\infty}\frac{(-2)^n}{n!}x^{n+4}$ 2 ; K.T. $=\mathbb{R}$ 1

$x_0=0$

$$g^{(100)}(0)=100!\cdot a_{100} = 100!\,\frac{(-2)^{96}}{96!} = \underline{\underline{100\cdot 99\cdot 98\cdot 97\cdot 2^{96}}} \quad 3$$

$n+4=100$

$n=96$

5, $\boxed{17}$ $f(x,y)=\dfrac{e^{y+2x}}{x^2+1}$; $P_0(0,1)$

a, $\boxed{9}$

$$f'_x=\frac{2e^{y+2x}(x^2+1)-e^{y+2x}\cdot 2x}{(x^2+1)^2} \quad 3$$

$$f'_y=\frac{e^{y+2x}}{x^2+1} \quad 2$$

Mindenütt léteznek és folytonosak, tehát mindenütt létezik grad f. 2

$$\operatorname{grad} f(P_0)=f'_x(P_0)\,\underline{i}+f'_y(P_0)\,\underline{j}=2e\,\underline{i}+e\,\underline{j}=\begin{bmatrix}2e\\ e\end{bmatrix} \quad 2$$

b, $\boxed{5}$ $\underline{e}\parallel -8\underline{i}+6\underline{j}$; $\underline{e}=\dfrac{1}{\sqrt{(-8)^2+6^2}}\begin{bmatrix}-8\\6\end{bmatrix}=\dfrac{1}{10}\begin{bmatrix}-8\\6\end{bmatrix}=\dfrac{1}{5}\begin{bmatrix}-4\\3\end{bmatrix}$ 1

$$\left.\frac{df}{d\underline{e}}\right|_{P_0}=\langle \operatorname{grad} f(P_0),\ \underline{e}\rangle \;2\; = 2e\cdot\left(-\frac{4}{5}\right)+e\left(\frac{3}{5}\right)=(-8+3)\frac{e}{5}=\underline{\underline{-e}} \quad 2$$

c, $\boxed{3}$

$$\max\left.\frac{df}{de}\right|_{P_0}=\left|\operatorname{grad} f(P_0)\right|=\sqrt{(2e)^2+e^2}=\underline{\underline{\sqrt{5}\,e}} \quad 3$$

6, $\boxed{9}$ $g(x,y) = f(3x^2+2y)$; $f \in C^2(\mathbb{R})$

$$g'_x(x,y) = f'(3x^2+2y)\cdot 6x \qquad (2)$$

$$g'_y(x,y) = f'(3x^2+2y)\cdot 2 \qquad (2)$$

$$g''_{xx}(x,y) = 6f'(3x^2+2y) + 36x^2 f''(3x^2+2y) \qquad (2)$$

$$g''_{yx}(x,y) \underset{\uparrow}{\overset{(1)}{=}} g''_{xy}(x,y) = 12x\, f''(3x^2+2y) \qquad (2)$$

Young-tétel

7, $\boxed{21}$

$$f(x,y) = \begin{cases} \dfrac{y^2(2x+3)}{x^2+y^2}, & \text{ha } (x,y) \neq (0,0) \\[2ex] 3, & \text{ha } (x,y) = (0,0) \end{cases}$$

a, Határérték, folytonosság az origóban?

$\boxed{8}$ $\left.\begin{array}{l} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{array}\right|$ $f(x,y) = \dfrac{\cancel{\rho^2}\sin^2\varphi\,(2\rho\cos\varphi+3)}{\cancel{\rho^2}\underbrace{(\sin^2\varphi+\cos^2\varphi)}_{1}} = 2\rho\sin^2\varphi\cos\varphi + 3\sin^2\varphi$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\rho\to 0}\left(2\rho\sin^2\varphi\cos\varphi + 3\sin^2\varphi\right) = 3\sin^2\varphi \quad \text{függ } \varphi\text{-től,}$$

tehát f-nek <u>nem létezik</u> a (6)
határértéke $\Rightarrow$ <u>nem folytonos</u> $(0,0)$-ban
(2)

$\boxed{4}$ b, $f'_y(x,y) = \dfrac{2y(2x+3)(x^2+y^2) - y^2(2x+3)\cdot 2y}{(x^2+y^2)^2}$, ha $(x,y)\neq(0,0)$ (4)

$\boxed{7}$ c, $f'_x(0,0) = \lim\limits_{h\to 0}\dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0}\dfrac{0-3}{h} = \nexists$ (4)

$f'_y(0,0) = \lim\limits_{h\to 0}\dfrac{f(0,h)-f(0,0)}{h} = \lim\limits_{h\to 0}\dfrac{\frac{3h^2}{h^2}-3}{h} = \lim\limits_{h\to 0}\dfrac{0}{h} = 0$ (3)

$\boxed{2}$ d, f nem diff.-ható totálisan az origóban, mert nem folyt., ill. $\nexists f'_x(0$

Pótfeladatok:

8, $\boxed{11}$

a, $f(x) = \frac{1}{x-7} = \frac{1}{(x-3)-4} = \frac{-1}{4}\,\frac{1}{1-\left(\frac{x-3}{4}\right)} = \sum_{n=0}^{\infty} \frac{-1}{4}\cdot\frac{1}{4^n}(x-3)^n$

$x_0 = 3$

$\boxed{6}$

$|q| = \left|\frac{x-3}{4}\right| < 1$, azaz $\underline{\underline{K.T. = (1,\ 7)}}$

b, $g(x) = \frac{1}{x^2-5} = \frac{-1}{5}\,\frac{1}{1-\frac{x^2}{5}} = \sum_{n=0}^{\infty} \frac{-1}{5}\,\frac{x^{2n}}{5^n}$

$\boxed{5}$ $x_0 = 0$

$|q| = \left|\frac{x^2}{5}\right| < 1$, azaz $\underline{\underline{K.T.:\ (-\sqrt{5},\ +\sqrt{5})}}$

9, $\boxed{9}$

$f(x,y) = 3x^2y + 4xy^4 - 2y + 15$; $x_0 = 1$, $y_0 = 0$

Érintősík?

$f(x_0, y_0) = 15$

$f'_x(x,y) = 6xy + 4y^2$; ② $f'_x(x_0,y_0) = 0$

$f'_y(x,y) = 3x^2 + 16xy^3 - 2$; ② $f'_y(x_0,y_0) = 3-2 = 1$

Általában: $z - f(x_0,y_0) = f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)$ ②

Most: $\underline{\underline{z - 15 = y}}$ ③