Analízis elmélet

2007. 09. 13.
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I. félév:
- Számsorozatok
- Számsorok
- Valós egyváltozós fv-k

1) $a_n = \dfrac{5-2n^2}{n^2-n+10}$ $A = -2$

$\sim -\dfrac{2n^2}{n^2}$ $|a_n - A| < \varepsilon$, ha $n > N(\varepsilon)$

$$|a_n - A| = \left|\frac{5-2n^2}{n^2-n+10}+2\right| = \left|\frac{5-2n^2+2n^2-2n+20}{n^2-n+10}\right| = \left|\frac{-2n+25}{n^2-n+10}\right| =$$

$$\underset{n\geq 13}{=} \frac{2n-25}{n^2-n+10} \le \frac{2n}{\frac{1}{2}n^2} = \frac{4}{n} < \varepsilon$$

meg kell nézni, hogy nullsorozat-e $n > \dfrac{4}{\varepsilon}$ $\left(\cancel{N(\varepsilon) = \left[\frac{4}{\varepsilon}\right]}\right)$

$N(\varepsilon) = \max\left\{\left[\frac{4}{\varepsilon}\right]; 12\right\}$

2) Konv.-e? $a_n = (-1)^n$ NEM $\varepsilon \geq 2$-re jó, $\varepsilon < 2$-re nem

$\Downarrow$

divergens, korlátos

3.) $a_n = 2n^5 - n^4 + 5n^2 + 9 \to \infty$ (D)-vel biz.-k!

$\forall P > 0$-hoz $\exists N(P)$, hogy $a_n > P$, ha $n > N(P)$

$a_n = 2n^5 - 4n^4 + 5n^2 + 9 > 2n^5 - n^5 + 0 + 0 > P$

$n^5 > P$

$n > \sqrt[5]{P}$ $N(P) = \left[\sqrt[5]{P}\right]$

4.) $b_n = 3n^2 - 2n^3 + 1 \to -\infty$

$\forall M < 0$ $\exists N(M)$: $b_n < M < 0$, ha $n > N(M)$

$b_n = \boxed{3n^2 - 2n^3 + 1 < M}$ (<0) ha $n > N(M)$

—"— $< \dots < M$ Nehézkes

$(-1)\cdot[\quad]$: $2n^3 \underbrace{-3n^2-1}_{-(3n^2+1)} > 2n^3 - n^3 > (-M) > 0 \Rightarrow n > \sqrt[3]{-M}$

$n > N_1$, $\exists N_1$

$N(M) = \max\{\dots\}$

Analízis

2007. 09. 18.
Kbuga I

Nagyságrendek összehasonlítása (n^n ; $n!$; 2^n ; $n^{(k)}$; $\log n$)

Ⓣ $\boxed{\begin{array}{ll} a)\ \frac{n^n}{n!} \to \infty & c)\ \frac{2^n}{n} \to \infty \\ b)\ \frac{n!}{2^n} \to \infty & d)\ \frac{n}{\log n} \to \infty \end{array}}$

$\frac{\infty}{\infty}$ határozatlan $\qquad \frac{n^2}{n^2} \to 1 \qquad \frac{n^2}{n} \to \infty \qquad \frac{n}{n^2} \to 0$

a) $\frac{n^n}{n!} = \frac{n}{1} \cdot \frac{n}{2} \cdot \frac{n}{3} \cdot \ldots \cdot \frac{n}{n-1} \cdot \frac{n}{n} \geq n \cdot 1 \cdot 1 \cdot \ldots \cdot 1 = n \to \infty$

⇓ spec. rendőrelv

$\frac{n^n}{n!} \to \infty$

Spec. rendőrelv: $a_n \geq b_n$
$\downarrow$
$\infty \Rightarrow a_n \to \infty$

b.) $\frac{n!}{2^n} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1}{2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 \cdot 2} \geq \frac{n}{2} \cdot 1 \cdot 1 \cdot \ldots \cdot 1 \cdot \frac{1}{2} = \frac{n}{4} \to \infty$

⇓ spec. rendőrelv

$\frac{n!}{2^n} \to \infty$

c) TB $c_n = \frac{2^n}{n}$

Igaz, hogy $c_{n+1} = \frac{2^{n+1}}{n+1} \geq \frac{2^n}{n}$ monoton nő

$c_{2n} \to \infty$

$\Big\} \Rightarrow c_n \to \infty$

Tehát $\boxed{n^n \gg n! \gg 2^n \gg n^k \gg n^{\frac{1}{k}} \gg \log n}$ $\quad n \in \mathbb{N}^+$

$(\sqrt[k]{n})$ $\qquad (k \in \mathbb{N}^+)$

↑ nagyságrendileg ~~...~~ erősebb

Ⓣ $a_n \to \infty \Rightarrow \frac{1}{a_n} \to 0$ $\qquad$ TB

Ⓣ $\boxed{\frac{\log n}{n} \to 0 \ ; \ \frac{n}{2^n} \to 0 \ ; \ \frac{2^n}{n!} \to 0 \ ; \ \frac{n!}{n^n} \to 0}$

—o—

Pl. $a_n = \frac{3n^2+4n+7}{n^2+n+1} \to 3$ $\quad N(\varepsilon)=?$

$|a_n - A| < \varepsilon$ ha $n > N(\varepsilon)$

$$\left|\frac{3n^2+4n+7}{n^2+n+1} - 3\right| = \left|\frac{3n^2+4n+7-3n^2-3n-3}{n^2+n+1}\right| = \frac{n+4}{n^2+n+1} \underset{\substack{n>4\\ n>3}}{<} \frac{n+n}{n^2+0+0} = \frac{2}{n} < \varepsilon$$

$$n > \frac{2}{\varepsilon} \Rightarrow N(\varepsilon) = \max\left\{\left[\frac{2}{\varepsilon}\right]; 4\right\}$$

Pl. $a_n = \frac{n^2 - 10^8}{5n^6 + 2n^3 - 1}$

a) $\lim a_n = ?$
b) $N(\varepsilon) = ?$

a) $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2 - 10^8}{5n^6 + 2n^3 - 1} = \lim_{n\to\infty} \underbrace{\frac{n^2}{n^6}}_{=\frac{1}{n^4} \to 0} \cdot \frac{1 - \frac{10^8}{n^2}}{5 + \frac{2}{n^3} - \frac{1}{n^6}} = 0 \cdot \frac{1-0}{5+0-0} = 0$$

$\frac{1}{n^\alpha} \to 0$, ha $\alpha > 0$

b) $$|a_n - A| = \left|\frac{n^2 - 10^8}{5n^6 + 2n^3 - 1} - 0\right| \underset{\substack{n \geq 10^4 \\ n > 10^4 - 1 \\ n > N_1}}{=} \frac{n^2 - 10^8}{5n^6 + \underbrace{2n^3 - 1}_{>0}} < \frac{n^2}{5n^6 + 0} = \frac{1}{5n^4} < \varepsilon$$

$n > \sqrt[4]{\frac{1}{5\varepsilon}}$ $\quad (n \in \mathbb{N})$

$N(\varepsilon) = \max\left\{\left[\sqrt[4]{\frac{1}{5\varepsilon}}\right]; 10^4\right\}$

HF: $a_n = \frac{2n^2+3n+6}{3n^2-n+1} \to \frac{2}{3}$ $\quad N(\varepsilon) = ?$

Pl. Vizsgálja konv. szempontjából az alábbi sorozatokat.

a) $$a_n = \frac{(n+1)!}{(5-2n)\,n!} = \frac{n+1}{-2n+5} = \frac{n}{-2n} \cdot \frac{1+\frac{1}{n}}{1-\frac{5}{2n}} = -\frac{1}{2}$$

b) $$a_n = \frac{\binom{n}{2}}{\binom{n}{3}} = \frac{\frac{n!}{2!\cdot(n-2)!}}{\frac{n!}{3!\cdot(n-3)!}} = \frac{6\cdot(n-3)!}{2\cdot(n-2)!} =$$

$$= 3\cdot\frac{(n-3)!}{(n-3)!\cdot(n-2)} = \frac{3}{n-2} = \frac{1}{n}\cdot\frac{3}{1-\frac{2}{n}} \to 0$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n\cdot(n-1)(n-2)\cdot\ldots\cdot(n-(k-1))}{1\cdot 2\cdot\ldots\cdot k}$$

$$\binom{8}{4} = \frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2\cdot1}$$

$$\binom{-1/2}{3} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)}{1\cdot 2\cdot 3}$$

Belátható, hogy

$$\boxed{\lim_{n\to\infty} a^n = \begin{cases} 0, \text{ ha } |a|<1 \\ 1, \text{ ha } a=1 \\ \infty, \text{ ha } a>1 \\ \text{oszcillálva divergens egyébként} \end{cases} \quad a\in\mathbb{R}}$$

$(a_n = a^n,\ a\in\mathbb{R}$

$a=-1 \qquad (-1)^n$ oszcillál $-1,\ 1$

$a=-2 \qquad (-2)^n = (-1)^n 2^n$ — n ps: 2^n, n pt: -2^n

Sőt $\quad n\cdot\left(\frac{1}{2}\right)^n \to 0 \qquad \frac{n}{2^n}$

$\infty\cdot 0$

Ált: $\boxed{\lim_{n\to\infty} n^k\cdot a^n = 0, \text{ ha } |a|<1,\ k\in\mathbb{N}^+}$

Pl $\quad a_n = \frac{n^3\cdot 2^n+3^n}{2^{2n}-3n^2} = \frac{n^3\cdot 2^n+3^n}{4^n-3n^2} = \frac{\overbrace{n^3\cdot\left(\frac{1}{2}\right)^n}^{k=3,\ a=\frac{1}{2}}+\overbrace{\left(\frac{3}{4}\right)^n}^{k=0,\ a=\frac{3}{4}}}{1-3\cdot n^2\cdot\left(\frac{1}{4}\right)^n} \xrightarrow[a=\frac{1}{4}]{k=2} \frac{0+0}{1+0}=0$

$\lim_{n\to\infty} n^k\cdot a^n = 0$, ha $|a|<1$, $k\geq 0$

Pl $\quad a_n = \frac{(-4)^n}{3^n+1} = (-1)^n\cdot\overbrace{\frac{4^n}{3^n+1}}^{d_n}$ $\quad$ Tehát $a_{2n}\to\infty$; $a_{2n+1}\to-\infty$ nagyon a sorozat divergens

$b_n = \frac{(-4)^n}{3^n+4^n} = (-1)^n\cdot\frac{4^n}{3^n+4^n} = (-1)^n\cdot\frac{1}{\left(\frac{3}{4}\right)^n+1} \to 1$ $\quad$ $\left.\begin{matrix} b_{2n}\to 1 \\ b_{2n+1}\to -1\end{matrix}\right\}$ divergens

$c_n = \frac{(-4)^n}{3^n+5^n} = \frac{\left(-\frac{4}{5}\right)^n \to 0}{\left(\frac{3}{5}\right)^n+1} \to \frac{0}{0+1} = 0$

$d_n = \frac{4^n}{3^n+1} \geq \frac{4^n}{3^n+3^n} = \frac{1}{2}\left(\frac{4}{3}\right)^n \to \infty$ $\quad$ felhasználtuk, hogy $\lim_{n\to\infty} a^n=\infty$, ha $|a|<1$

$a=\frac{4}{3}>1$

HF. $\quad a_n = \frac{2^{2n}}{(-3)^n+q^n}$ $\quad q\in\mathbb{R}$ fv-ében vizsgálni a_n-t.

(R2) $a_n = \sqrt{2n^2+5n} - \sqrt{2n^2-n} = \frac{2n^2+5n-2n^2+n}{\sqrt{2n^2+5n}+\sqrt{2n^2-n}} = \frac{6n}{\sqrt{n^2}\left(\sqrt{2+\frac{5}{n}}+\sqrt{2-\frac{1}{n}}\right)}$

$= \frac{6n}{\sqrt{n^2}} \cdot \frac{\cancel{6}}{\sqrt{2}+\sqrt{2}} = \frac{6}{2\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$

$\frac{n}{|n|}$

(T) $a_n \to A \geq 0 \Rightarrow \sqrt{a_n} \to \sqrt{A}$

HF: $a_n = \sqrt{n^4+2n^2+3} - \sqrt{n^4+n}$

Analízis elmélet

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Pl. $a_n = \frac{n^2-n^3}{2n^3+5n+3} = \frac{-n^3}{n^3}\cdot\frac{1-\frac{1}{n}}{2+\frac{5}{n^2}+\frac{3}{n^3}} \to -1\cdot\frac{1}{2} = -\frac{1}{2}$

$b_n = \frac{n^2-5n}{n^3+3} = \frac{n^2}{n^3}\cdot\frac{1-\frac{5}{n}}{1+\frac{3}{n^3}} \to 0\cdot\frac{1}{1} = 0$

$c_n = \frac{(n^2+2)^2}{2n^4+7n^3+1} = \frac{n^4+4n^2+4}{2n^4+7n^3+1} = \frac{n^4}{n^4}\cdot\frac{1+\frac{4}{n^2}+\frac{4}{n^4}}{2+\frac{7}{n}+\frac{1}{n^4}} \to 1\cdot\frac{1}{2} = \frac{1}{2}$

$d_n = \frac{n^3+8}{2n^2+9} = \frac{n^3}{n^2}\cdot\frac{1+\frac{8}{n^3}}{2+\frac{9}{n^2}} \to \infty$

(T: Spec. rendőrelv: $a_n \geq b_n \to \infty \Rightarrow a_n \to \infty$)

$d_n = \frac{n^3+8}{2n^2+9} \geq n\cdot\frac{1}{2+9} = \frac{1}{11}\cdot n \to \infty \Rightarrow d_n \to \infty$

$d_n = n\cdot \underset{\downarrow \frac{1}{2}}{u_n} > n\cdot\frac{1}{4} \to \infty \Rightarrow d_n \to \infty$

$u_n \to \frac{1}{2} \Rightarrow \frac{1}{2}-\varepsilon < u_n < \frac{1}{2}+\varepsilon$, ha $n > N_1$

$\varepsilon = \frac{1}{4}$

$a_n\cdot b_n \to A\cdot B$.

$\left(1+\frac{1}{n}\right)^3 = (\;)\cdot(\;)\cdot(\;) = 1$ (mindegyik $\to 1$)

$\left(1+\frac{1}{n}\right)^k \to 1 \quad k \in \mathbb{N}^+$ konstans

De $\left(1+\frac{1}{n}\right)^n \not\to 1$

—— o ——

Pl. $a_n = \sqrt{\frac{4n^2+6n+1}{8n^2+5}} = \sqrt{\frac{n^2}{n^2}\cdot\frac{4+\frac{6}{n}+\frac{1}{n^2}}{8+\frac{5}{n^2}}} = \sqrt{1\cdot\frac{1}{2}} = \frac{\sqrt{2}}{2}$

$b_n = \frac{\sqrt{4n^2+6n+1}}{\sqrt[3]{8n^3+3n^2+5}} = \frac{\sqrt{4n^2}}{\sqrt[3]{8n^3}}\cdot\frac{\sqrt{1+\frac{6}{n}+\frac{1}{n^2}}}{\sqrt[3]{1+\frac{3}{8n}+\frac{5}{8n^3}}} \to \frac{2n}{2n}\cdot 1 = 1$

Pl. $a_n = \sqrt{n^4+1} - \sqrt{n^4+2n^2+3} = \frac{n^4+1-n^4-2n^2-3}{\sqrt{n^4+1}+\sqrt{n^4+2n^2+3}} = \frac{-2n^2-2}{\sqrt{n^4}\cdot\left(\sqrt{1+\frac{1}{n^4}}+\sqrt{1+\frac{2}{n^2}+\frac{3}{n^4}}\right)} = \frac{n^2}{n^2}\cdot\frac{-2-\frac{1}{n^2}}{1+1} \to -1$

Analízis elmélet

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Pl. $a_n = (c^2-2)^n$ Milyen c-re konvergens?

$$\lim_{n\to\infty} a^n = \begin{cases} 0, & \text{ha } |a|<1 \\ 1, & \text{ha } a=1 \\ \infty, & \text{ha } a>1 \\ \text{oszc. div}, & \text{ha } a\le -1 \end{cases}$$

$c^2-2=1$

$c^2=3 \Rightarrow |c|=\sqrt{3} \Rightarrow c=\sqrt{3} \vee c=-\sqrt{3} \Rightarrow a_n \to 1$

$-1 < c^2-2 < 1 \Rightarrow 1 < c^2 < 3 \Rightarrow 1<|c|<\sqrt{3} \Rightarrow c\in(-\sqrt{3};-1)\cup(1;\sqrt{3}) \Rightarrow a_n \to 0$

Pl. $a_n = \dfrac{4^{n+1}+(-3)^{n-1}}{2^{2n}+2^{n+2}} = \dfrac{4\cdot 4^n + \frac{(-3)^n}{3}}{4^n+4\cdot 2^n} = \dfrac{4+\frac{1}{3}\cdot\left(-\frac{3}{4}\right)^n}{1+4\cdot\left(\frac{2}{4}\right)^n} \to \dfrac{4}{1}=4$

Pl. $a_n = \dfrac{(-2)^{2n+3}+(-3)^n}{n^2\cdot 4^{n+2}+4^{2n+1}} = \dfrac{-8\cdot 4^n+(-3)^n}{16\cdot n^2\cdot 4^n+4\cdot 16^n} = \dfrac{-8\cdot\left(\frac{4}{16}\right)^n+\left(\frac{-3}{16}\right)^n}{16\, n^2\cdot\left(\frac{4}{16}\right)^n+4} \to 0$

$$\lim_{n\to\infty} n^k\cdot a^n = 0, \text{ ha } |a|<1,\ k\in\mathbb{N}$$

Pl. $a_n = \dfrac{(-4)^n+3^n}{2^{2n}+2^n} = \dfrac{4^n}{4^n}\cdot\dfrac{(-1)^n+\left(\frac{3}{4}\right)^n}{1+\left(\frac{1}{2}\right)^n}$

- n páros: $a_n = \dfrac{1+\left(\frac{3}{4}\right)^n}{1+\left(\frac{1}{2}\right)^n} \to 1$
- n ptlan: $a_n = \dfrac{-1+\left(\frac{3}{4}\right)^n}{1+\left(\frac{1}{2}\right)^n} \to -1$

$a_{2n}\to 1 \quad a_{2n+1}\to -1 \Rightarrow a_n$ divergens (2 torlódási pont)

Pl. $a_n = \dfrac{2^{3n}+3^n}{1+5^{n+1}} = \left(\dfrac{8^n+3^n}{1^n+5\cdot 5^n} = \dfrac{1+\left(\frac{3}{8}\right)^n}{\left(\frac{1}{8}\right)^n+5\cdot\left(\frac{5}{8}\right)^n}\right)$

$\ge \dfrac{8^n}{5^n+5\cdot 5^n} = \dfrac{1}{6}\cdot\left(\dfrac{8}{5}\right)^n \to \infty \Rightarrow a_n\to\infty$

Pl. $a_n = \sqrt[n]{4n} = \sqrt[n]{4}\cdot\sqrt[n]{n} \to 1$

Pl. $a_n = \sqrt[4n]{4n} \to 1$, mert $\sqrt[n]{n}$ részsorozata

T. $a_n \to A \Rightarrow \forall\, a_{n_i} \to A$

(a_{n_i}) részsorozata

(Pl.) $a_n = \sqrt[4n]{n} = \sqrt[4n]{4n \cdot \frac{1}{4}} = \underbrace{\sqrt[4n]{4n}}_{\sqrt[n]{n} \text{ részsorozata}} \cdot \underbrace{\sqrt[4n]{\frac{1}{4}}}_{\sqrt[n]{p} \text{ részsorozata}} \to 1 \cdot 1 = 1$

vagy $a_n = n^{\frac{1}{4n}} = (n^{\frac{1}{n}})^{\frac{1}{4}} = \sqrt[4]{\sqrt[n]{n}} \to \sqrt[4]{1} = 1$

(Pl.) $a_n = \sqrt[n]{4n+3}$

$$\underset{\downarrow 1}{\sqrt[n]{3}} \le a_n = \sqrt[n]{4n+3} \le \sqrt[n]{\underbrace{4n+3n}_{7n}} = \underset{\downarrow 1}{\sqrt[n]{7}} \cdot \underset{\downarrow 1}{\sqrt[n]{n}} \to 1$$

⇓ rendőrelv

$a_n \to 1$

(Pl.) $a_n = \sqrt[n]{\frac{8n^7+3n+1}{n^5+5}}$

$$1 \leftarrow \frac{1}{\underset{\downarrow 1}{\sqrt[n]{6}} \cdot \underset{\downarrow 1}{\sqrt[n]{n^5}}} = \sqrt[n]{\frac{8n^7 \cdot 1}{\underbrace{n^5+5n^5}_{\frac{1}{6n^5}}}} \le a_n = \sqrt[n]{\frac{8n^7+3n+1}{n^5+5}} \le \sqrt[n]{\frac{8n^7+3n^7+n^7}{1}} = \sqrt[n]{\frac{12}{5}n^7} = \underset{\downarrow 1}{\sqrt[n]{\frac{12}{5}}} \cdot \underset{\downarrow 1}{(\sqrt[n]{n})^7} \to$$

$a_n \to 1$

(Pl.) $a_n = \sqrt[n]{\frac{8n^3+3}{n^3+2n+5}}$ $\qquad b_n = \sqrt[3]{\frac{8n^3+3}{n^3+2n+5}}$ HF

$$\cancel{a_n = \sqrt[n]{\frac{n^3}{n^3} \cdot \frac{8+\frac{3}{n^3}}{1+\frac{2}{n^2}+\frac{5}{n^3}}} \to \sqrt[n]{8} \to 1}$$

ÍGY NEM

Ez „letakarás".

(elfelejtem közben a határokat)

Rossz példa $\cancel{(1+\frac{1}{n})^n \to 1}$ „letakarással"

$a_n = \sqrt[n]{c_n}$ $\qquad c_n = \frac{8n^3+3}{n^3+2n+5} \to 8 \Rightarrow 0 < 8-\varepsilon \le c_n \le 8+\varepsilon$, ha $n > N$

$7 < c_n < 9$ $(\varepsilon=1)$, ha $n > N(1)$

Ha $n > N(1)$: $\underset{\downarrow 1}{\sqrt[n]{7}} < a_n = \sqrt[n]{c_n} < \underset{\downarrow 1}{\sqrt[n]{9}}$

$\Rightarrow a_n \to 1$

Másik megoldás

$$\underset{\downarrow 1}{\frac{1}{\sqrt[n]{8} \cdot (\sqrt[n]{n})^3}} = \sqrt[n]{\frac{1}{n^3+2n^3+5n^3}} \le a_n = \sqrt[n]{\frac{8n^3+3}{n^3+2n+5}} \le \sqrt[n]{\frac{8n^3+3n^3}{1}} = \underset{\downarrow 1}{\sqrt[n]{11}} \cdot \underset{\downarrow 1}{\sqrt[n]{n^3}} \to 1$$

$\Rightarrow a_n \to 1$

Analízis elmélet

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Rekurzív sorozatok

(Pl) $a_1 = 7$ $\qquad n = 1, 2, \ldots$

$a_{n+1} = 7 - \frac{6}{a_n}$ $\qquad (a_n) = (7, 6.1428, 6.02, \ldots)$

Konvergens-e? Ha igen, $A = ?$

Ha konvergens, akkor $\underbrace{\lim a_{n+1}}_{A} = \lim \left(7 - \frac{6}{a_n}\right) = A = 7 - \frac{6}{A}$

$A^2 - 7A + 6 = 0$

$(A-1)(A-6) = 0$ $\quad \begin{cases} A = 1 \\ A = 6 \end{cases}$

Sejtés: $a_n > 6$

Biz: Teljes indukció:
1) $a_1 > 6$ ✓
2) Tfh $a_n > 6$
3) Igaz-e: $a_{n+1} > 6$

$a_{n+1} = 7 - \frac{6}{a_n} \overset{?}{>} 6$

2.) miatt $a_n > 6 > 0$

$\frac{1}{a_n} < \frac{1}{6} \quad / \cdot (-6)$

$-\frac{6}{a_n} > -1 \quad / +7$

$7 - \frac{6}{a_n} > 6$ ✓

Sejtés: $(a_n) \searrow$

Biz: Teljes indukció:
1) $a_1 > a_2 > a_3$ ✓
2) Tfh $a_{n-1} > a_n$
3) Igaz-e? $\boxed{7 - \frac{6}{a_{n-1}}} = a_n \overset{?}{>} a_{n+1} = \boxed{7 - \frac{6}{a_n}}$

Tudjuk: $a_{n-1} > a_n > 6 > 0$

$\frac{1}{a_{n-1}} < \frac{1}{a_n}$

$-\frac{6}{a_{n-1}} > -\frac{6}{a_n}$

$7 - \frac{6}{a_{n-1}} > 7 - \frac{6}{a_n}$

$a_n > a_{n+1}$ ✓

(T) Az (a_n) monoton és korlátos $\Rightarrow$ $\overset{(a_n)}{\text{konvergens}}$

$\lim a_n = 6$

Pl $a_{n+1} = 12 + \frac{20}{a_n}$ $a_1 = 4$ $(a_n) = (4, 7, 9, 14, ...)$

a) Mutassa meg, hogy $2 \le a_n \le 10$

b.) Ig., hogy (a_n) konvergens! $A = 2$

a) Teljes ind. 1.) $2 \le a_i \le 10$ $i = 1, (2, 3)$
2.) Tfh $2 \le a_n \le 10$
3.) Igaz-e? $2 \overset{?}{\le} a_{n+1} = 12 - \frac{20}{a_n} \overset{?}{\le} 10$

A 2.) miatt: $0 < 2 \le a_1 \le 10$

$$\frac{1}{2} \ge \frac{1}{a_1} \ge \frac{1}{10} \quad \Big| \cdot (-20)$$

$$-10 \le -\frac{20}{a_1} \le -2 \quad \Big| + 12$$

$$2 \le 12 - \frac{20}{a_n} = a_{n+1} \le 10 \quad ✓$$

b.) Segítés: $(a_n) \nearrow$

Biz: T. I. 1.) $a_1 < a_2 (< a_3)$
2.) Tfh $a_{n-1} < a_n$
3.) Igaz-e: $a_n = \boxed{12 - \frac{20}{a_{n-1}} \overset{?}{<} 12 - \frac{20}{a_n}} = a_{n+1}$

A 2.) miatt: $0 < 2 \le a_{n-1} < a_n$

$$\frac{1}{a_{n-1}} > \frac{1}{a_n}$$

$$-\frac{20}{a_{n-1}} < -\frac{20}{a_n}$$

$$a_n = 12 - \frac{20}{a_{n-1}} < 12 - \frac{20}{a_n} = a_{n+1}$$

Az (a_n) monoton nő és felülről korlátos $\Rightarrow$ (a_n) konv

$$A = 12 - \frac{20}{A}$$

$$A^2 - 12A + 20 = 0 \quad \begin{cases} A_1 = 2 & ⚡ \\ \boxed{A_2 = 10} & ✓ \end{cases}$$

2007. 09. 26.
Kónya

Analízis elmélet

$e_n = \left(1+\frac{1}{n}\right)^n \nearrow$ és $e_n < 3 \Rightarrow (e_n)$ konv $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

Pl $a_n = \left(1+\frac{1}{n^2}\right)^{n^2} \to e$, mert (e_n) részsorozata

Pl $a_n = \left(1+\frac{1}{n^2}\right)^{2n^2} = \Big(\underbrace{\left(1+\frac{1}{n^2}\right)^{n^2}}_{\to e}\Big)^2 \to e^2$

Pl $a_n = \left(1+\frac{1}{2n^2}\right)^{n^2} = \sqrt{\underbrace{\left(1+\frac{1}{2n^2}\right)^{2n^2}}_{\to e}} \to e^{\frac{1}{2}}$

Tehát $\lim\limits_{n\to\infty}\left(1+\frac{1/2}{n^2}\right)^{n^2} \to e^{\frac{1}{2}}$

Pl $\left(1+\frac{1}{3n^2}\right)^{2n^2} = \Big(\underbrace{\left(1+\frac{1}{3n^2}\right)^{3n^2}}_{\to e}\Big)^{\frac{2}{3}} \to e^{\frac{2}{3}}$

Belátható, hogy $\boxed{\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x}$ TB

Pl $a_n = \underbrace{\left(\frac{n+5}{n+3}\right)^n}_{1^\infty} = \frac{\left(1+\frac{5}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n} \to \frac{e^5}{e^3} = e^2$ $\left(1+\frac{\square}{\bigcirc}\right)^{\bigcirc} \to e^{\square}$

Vagy: $a_n = \left(\frac{n+3+2}{n+3}\right)^n = \left(1+\frac{2}{n+3}\right)^n = \left(1+\frac{2}{n+3}\right)^{n+3}\cdot\left(1+\frac{2}{n+3}\right)^{-3} \to e^2\cdot 1^3 = e^2$

Pl $a_n = \left(\frac{4n+2}{4n+5}\right)^{2n} = \frac{\left(1+\frac{2}{2n}\right)^{2n}}{\sqrt{\left(1+\frac{5}{4n}\right)^{4n}}} \to \frac{e}{e^{5/2}} = e^{-\frac{3}{2}}$

$= \frac{\left(\left(1+\frac{1/2}{n}\right)^n\right)^2}{\left(\left(1+\frac{5/4}{n}\right)^n\right)^2} \to \frac{e^{\frac{1}{2}\cdot 2}}{e^{5/4\cdot 2}} = \frac{e}{e^{5/2}} = e^{-\frac{3}{2}}$

Pl $\left(\frac{2n^3-5}{2n^3+1}\right)^{n^3} = \frac{\left(1+\frac{-5}{2n^3}\right)^{n^3}}{\left(1+\frac{1}{2n^3}\right)^{n^3}} = \frac{\left(1+\frac{-5/2}{n^3}\right)^{n^3}}{\left(1+\frac{1/2}{n^3}\right)^{n^3}} = \frac{e^{-5/2}}{e^{1/2}} = e^{-3}$

Pl $\underbrace{a_n}_{1^\infty} = \left(\frac{4n+2}{4n+7}\right)^n = \frac{\left(1+\frac{2}{4n}\right)^n}{\left(1+\frac{7}{4n}\right)^n} = \frac{\left(1+\frac{2/4}{n}\right)^n}{\left(1+\frac{7/4}{n}\right)^n} \to \frac{e^{2/4}}{e^{7/4}} = e^{-5/4}$

$\underbrace{b_n}_{\left(\frac{2n}{4n}\right)^n} = \left(\frac{2n+2}{4n+7}\right)^n$ $\underbrace{0}_{\to 0} < b_n \underset{n\geq N_1}{<} \left(\frac{2n+n}{4n}\right)^n = \underbrace{\left(\frac{3}{4}\right)^n}_{\to 0}$

$\Rightarrow b_n \to 0$

$$b_n = \left(\frac{3n+1}{4n+7}\right)^n \qquad \underset{\to 0}{0} < b_n \underset{n>2}{<} \left(\frac{3n+\frac{n}{2}}{4n}\right)^n = \left(\frac{\frac{7}{2}n}{4n}\right)^n = \left(\frac{7}{8}\right)^n \to 0 \Rightarrow b_n \to 0$$

$$\underset{\left(\frac{4n}{2n}\right)^n}{c_n} = \left(\frac{4n+1}{2n+3}\right)^n < \cancel{\left(\frac{4n+4n}{2n}\right)^n = \left(\frac{8n}{2n}\right)^n = 4^n \to \infty \Rightarrow c_n \to \infty}$$

$$\underset{n>N_3}{\geq} \left(\frac{4n+1}{2n+n}\right)^n = \left(\frac{4}{3}\right)^n \to \infty \quad (a^n \to \infty, \text{ ha } a>1) \Rightarrow c_n \to \infty$$

Pé! $a_n = \left(1+\frac{1}{n^3}\right)^{n^3} \to e$, mert (e_n) részsorozata

$$b_n = \left(1+\frac{1}{n^3}\right)^{n^2} = \sqrt[n]{\left(1+\frac{1}{n^3}\right)^{n^3}} = \sqrt[n]{a_n} \qquad a_n \to e$$

$e-0{,}1 < a_n < e+0{,}1$, ha $n > N_1$

$$\underset{\to 1}{\sqrt[n]{e-0{,}1}} < b_n = \sqrt[n]{a_n} < \underset{\to 1}{\sqrt[n]{e+0{,}1}} \Rightarrow b_n \to 1$$

$$c_n = \left(1+\frac{1}{n^2}\right)^{n^3} = \left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)^n \geq 2^n \to \infty \Rightarrow c_n \to \infty$$

(e_n) részsorozata $(2 \leq e_n < 3)$

Analízis elmélet

2007.09.27
Kónya

(Pl) $a_n=\left(\frac{2n^2+9}{2n^2+4}\right)^{4n^2+5}=\left(\frac{\left(1+\frac{9/2}{n^2}\right)^{n^2}}{\left(1+\frac{4/2}{n^2}\right)^{n^2}}\right)^4\cdot\frac{\left(1+\frac{9}{2n^2}\right)^5}{\left(1+\frac{4}{2n^2}\right)^5}\to\left(\frac{e^{9/2}}{e^{4/2}}\right)^4\cdot 1^5=e^{10}$

(1^{∞})

(Pl) $a_n=\left(\frac{3n+1}{5n+9}\right)^n=\left(\frac{3n\cdot\left(1+\frac{1}{3n}\right)}{5n\cdot\left(1+\frac{9}{5n}\right)}\right)^n=\left(\frac{3}{5}\right)^n\cdot\frac{\left(1+\frac{1/3}{n}\right)^n}{\left(1+\frac{9/5}{n}\right)^n}\to 0\cdot\frac{e^{1/3}}{e^{9/5}}=0$

(Pl) $a_n=\left(\frac{n^4+1}{n^4+4}\right)^{n^3}=\sqrt[n]{\left(\frac{n^4+1}{n^4+4}\right)^{n^4}}:=\sqrt[n]{b_n}$

1^{∞}

$b_n=\frac{\left(1+\frac{1}{n^4}\right)^{n^4}}{\left(1+\frac{4}{n^4}\right)^{n^4}}\to\frac{e^1}{e^4}=e^{-3}=B>0$

$0<\underbrace{\frac{1}{e^3}-10^{-3}}_{:=p_1\in\mathbb{R}}<b_n<\underbrace{\frac{1}{e^3}+10^{-3}}_{:=p_2\in\mathbb{R}>0}$

$\underbrace{\sqrt[n]{p_1}}_{\to 1}<\sqrt[n]{b_n}<\underbrace{\sqrt[n]{p_2}}_{\to 1}$

$\left(\lim_{n\to\infty}\sqrt[n]{p}=1\ ,\ \text{ha } p>0\right)$

— 0 —

(Pl) $a_n=(-1)^n\cdot\left(5+\frac{2}{n}\right)$ $\qquad A=\{a_n : n\in\mathbb{N}^{+}\}$

$\overline{\lim}\ a_n=?$ $\qquad \underline{\lim}\ a_n=?$ $\qquad \sup A=?$ $\qquad \inf=?$

n ps: $a_n=5+\frac{2}{n}$ $\qquad$ 5, 5,5, 6 ; a_4, a_2 $\qquad a_{2k}\to 5$

$a_{2k}=5+\frac{2}{2k}$

n pt: $a_n=-\left(5+\frac{2}{n}\right)$ $\qquad$ a_1, a_3 ; -7, -6, -5 $\qquad a_{2k+1}\to -5$

$S=\{-5;5\}$ torlódási pontok

$\overline{\lim}\ a_n=\limsup\ a_n=5$

$\underline{\lim}\ a_n=\liminf\ a_n=-5$

$\sup A=6$

legkisebb felső korlát

$\inf A=-7$

legnagyobb alsó korlát

(Pl) $a_n = \underbrace{\frac{2n^4+5n^2+3}{n^4+2n+1}}_{:=b_n} \cdot \sin\frac{n\pi}{2}$ $\overline{\lim}\, a_n = ?$ $\underline{\lim}\, a_n = ?$ $\lim\limits_{n\to\infty} a_n = ?$

$b_n = \frac{n^4}{n^4} \cdot \frac{2+\frac{5}{n^2}+\frac{3}{n^4}}{1+\frac{2}{n^3}+\frac{1}{n^4}} \to 2$

Ha $n = 4k+1 \Rightarrow \sin n\frac{\pi}{2} = 1$ $a_n = b_n \to 2$

Ha $n = 2k \Rightarrow \sin n\frac{\pi}{2} = 0$ $a_n \to 0$

Ha $n = 4k+3 \Rightarrow \sin n\frac{\pi}{2} = -1$ $a_n \to -2$

$S = \{-2; 0; 2\}$

$\limsup a_n = 2$

$\underline{\lim}\, a_n = -2$

$\lim\limits_{n\to\infty} a_n \nexists$ (mert több torlódási pont van)

(Pl) Lehet-e $\overline{\lim}\, a_n = \infty$?

Igen, pl.: $a_n = -n$ $S = \{-\infty\}$ $\overline{\lim}\, a_n = -\infty$
$\underline{\lim}\, a_n = -\infty$

(Pl) $a_n = \left(\frac{2-n}{3+n}\right)^n = \left((-1) \cdot \frac{n-2}{n+3}\right)^n = (-1)^n \cdot \underbrace{\left(\frac{n-2}{n+3}\right)^n}_{b_n}$

$b_n = \frac{\left(1+\frac{-2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n} \to \frac{e^{-2}}{e^3} = \frac{1}{e^5}$

$a_{2k} \to \frac{1}{e^5}$ $S = \left\{\frac{-1}{e^5}; \frac{1}{e^5}\right\}$

$a_{2k+1} \to \frac{-1}{e^5}$ $\underline{\lim}\, a_n$ $\overline{\lim}\, a_n$

(Pl) $a_n = \frac{2^{2n}+(-4)^n}{4^{n+1}+5} = \frac{4^n+(-1)^n \cdot 4^n}{4 \cdot 4^n+5}$ n ps: $a_n = \frac{2 \cdot 4^n}{4 \cdot 4^n + 5} = \frac{4^n}{4^n} \cdot \frac{2}{4+\frac{5}{4^n}} \to \frac{2}{4} = \overline{\lim}$

$b_n = \frac{2^{2n}+(-4)^n}{5^n+6}$ n pt: $a_n = 0 \to 0 = \underline{\lim}\, a_n$

$b_n = \frac{4^n+(-4)^n}{5^n+6} = \frac{\left(\frac{4}{5}\right)^n + \left(-\frac{4}{5}\right)^n}{1+6\cdot\left(\frac{1}{5}\right)^n} \to \frac{0+0}{1+0} = 0 = \lim\limits_{n\to\infty} b_n = \overline{\lim}\, b_n = \underline{\lim}\, b_n$

<u>Numerikus sorok</u>

(Pl) $\sum\limits_{n=2}^{\infty} \frac{(-4)^{n+2}}{2^{3n+1}} = \sum\limits_{n=2}^{\infty} \frac{16 \cdot (-4)^n}{2 \cdot 8^n} = 8 \cdot \sum\limits_{n=2}^{\infty} \left(-\frac{1}{2}\right)^n = 8 \cdot \frac{\left(-\frac{1}{2}\right)^2}{1-\left(-\frac{1}{2}\right)} = 8 \cdot \frac{\frac{1}{4}}{\frac{3}{2}} = \frac{4}{3}$

(T) $\sum\limits_{1}^{\infty} a_k = S_a \in \mathbb{R}$ és $\sum\limits_{1}^{\infty} b_k = S_b \in \mathbb{R} \Rightarrow \sum\limits_{1}^{\infty} (a_k + b_k) = s_a + s_b$

Analízis elmélet

2007. 10. 03
Kónya

Numerikus sorok

Pl $\sum_{2}^{\infty} \frac{2}{k^2-1} = ? = \sum_{2}^{\infty} \frac{2}{(k-1)(k+1)}$

$\frac{2}{(k-1)(k+1)} = \frac{A}{k-1} + \frac{B}{k+1} \quad / (k-1)(k+1)$

Mindig egyértelműen $\exists$ A és B.

$2 = A(k+1) + B(k-1)$ *

minden k-ra igaz legyen
Praktikus helyettesítés: a régi nev. gyökei

$k := 1 \quad 2 = 2A$
$k := -1 \quad 2 = -2B$ $\Rightarrow A = 1, \; B = -1$

Vagy: $\left(\begin{array}{ll} k:=0 & 2 = A - B \\ k:=2 & 2 = 3A + B \end{array}\right)$

* bal és jobb oldal is k-ban polinom $\Rightarrow$ együtthatói egyenlők
$\forall$ k-ra u.az az helyettesítési értéke

$2 = \underbrace{k(A+B) + (A-B)}$

$\begin{cases} A - B = 2 \\ A + B = 0 \end{cases} \; \ldots$

D-vel $s = ?$

$s_n = \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k+1}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \ldots + \left(\cancel{\frac{1}{n-3}} - \cancel{\frac{1}{n-1}}\right) + \left(\cancel{\frac{1}{n-2}} - \frac{1}{n}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right)$

($k = n-2$, $k = n$)

$s_n = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} \to 1 + \frac{1}{2} = \frac{3}{2} = s$

Pl $\sum_{k=1}^{\infty} \frac{(-2)^{k+1} + 2^{2k-1}}{3^{2k+1}} = \sum_{k=1}^{\infty} \frac{-2 \cdot (-2)^k + 4^k \cdot \frac{1}{2}}{3 \cdot 9^k} \overset{T}{=} -\frac{2}{3} \sum_{k=1}^{\infty} \left(\frac{-2}{9}\right)^k + \frac{1}{6} \cdot \sum_{1}^{\infty} \left(\frac{4}{9}\right)^k =$

$q = -\frac{2}{9}$, $|q| < 1$ $\qquad q = \frac{4}{9}$, $|q| < 1$

$\text{ért} = -\frac{2}{3} \cdot \frac{-\frac{2}{9}}{1-\left(-\frac{2}{9}\right)} + \frac{1}{6} \cdot \frac{\frac{4}{9}}{1-\frac{4}{9}}$

Pl Konv-e?

a) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt[7]{n+2}} =$

$\cos n\pi = (-1)^n$ $\qquad$ Leibniz sor, ha $c_n \searrow 0$

$c_n = \frac{1}{\sqrt[7]{n+2}}$ $\qquad c_n \to 0$

$c_n \searrow$ triv. igaz.

Tehát L. sor $\Rightarrow$ konvergens $\Rightarrow s \approx s_{1000}$

$|H| = |s - s_{1000}| < c_{1001} = \frac{1}{\sqrt[7]{1001+2}}$

b.) $\sum_1^{\infty} (-1)^{n+1} \underbrace{\frac{1}{\sqrt[n]{n^3+8n^2}}}_{c_n}$

$c_n = \frac{1}{\sqrt[n]{n^3+8n^2}} \to 1 \neq 0$ (rendőrelv !!!) $\Rightarrow$ a sor divergens, mert nem teljesül a konvergencia szükséges feltétele.

c) $\sum_{n=1}^{\infty} (-1)^n \cdot \underbrace{\frac{3+\frac{1}{n}}{n+3}}_{=:c_n}$

$c_n := \frac{3+\frac{1}{n} \to 0}{n+3 \searrow \infty} \to 0 \quad \left(\frac{\text{konst}}{\infty} \to 0\right)$

$c_n \searrow$ (számláló $\Downarrow$, nev $\nearrow$ $\Rightarrow$ tört csökk)

$\Bigg\}$ Tehát Leibniz sor $\Rightarrow$ konv

(HF) a) $\sum_1^{\infty} (-1)^n \left(\frac{3n^2+5}{3n^2+2}\right)^{4n^2}$ konv-e?

b) $\sum_1^{\infty} (-1)^{n+1} \frac{n+1}{3n^2+8}$

$c_n \searrow \qquad c_{n+1} < c_n$

$\frac{(n+1)+1}{3(n+1)^2+8} < \frac{n+1}{3n^2+8}$

Analízis elmélet

2007.10.04

Konzultáció: 6. hét kedd 17⁰⁰

Kónya

Konvergensek?

a) $\frac{1}{2}-\frac{1}{3}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{3}\right)^2+\dots+\left(\frac{1}{2}\right)^{n}-\left(\frac{1}{3}\right)^{n}+\dots$ $\neq \sum_{n=1}^{\infty}\left(\left(\frac{1}{2}\right)^n-\left(\frac{1}{3}\right)^n\right)$!!

$S_{2n}=\frac{1}{2}-\frac{1}{3}+\dots+\left(\frac{1}{2}\right)^n-\left(\frac{1}{3}\right)^n=\frac{1}{2}+\dots+\left(\frac{1}{2}\right)^n-\left(\frac{1}{3}+\dots+\left(\frac{1}{3}\right)^n\right)=\underbrace{\frac{1}{2}\cdot\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}}_{\to s_1}-\underbrace{\frac{1}{3}\cdot\frac{1-\left(\frac{1}{3}\right)^n}{1-\frac{1}{3}}}_{\to s_2}$ $\to s_1-s_2$

$S_{2n+1}=S_{2n}+\underbrace{\left(\frac{1}{2}\right)^{n+1}}_{\to 0}\to s_1-s_2$

⇓

a sor konv.

Nem $\mathcal{L}$-sor, nem mon ↘

b) $1-\frac{2}{3}+\frac{1}{2}-\left(\frac{2}{3}\right)^2+\frac{1}{3}-\left(\frac{2}{3}\right)^3+\dots+\frac{1}{n}-\left(\frac{2}{3}\right)^n+\dots$

$s_{2n}=\dots=\underbrace{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}_{\sum_1^\infty \frac{1}{n}\text{ n-edik részletösszege}\ \downarrow\ \infty}-\underbrace{\left(\frac{2}{3}+\left(\frac{2}{3}\right)^2+\dots+\left(\frac{2}{3}\right)^n\right)}_{q=\frac{2}{3}\ \to s}\to\infty$ a sor divergens

Konv-e?

a) $\sum_1^\infty \frac{2n+3}{n^2+8}$ b) $\sum_1^\infty\left(\frac{2n+3}{n^2+8}\right)^2$ c) $\sum_1^\infty\left(\frac{2n+3}{n^2+8}\right)^{\frac{1}{n}}$

Tudjuk: $\sum_{n=0}^{\infty} a q^n=\frac{a}{1-q}$, ha $|q|<1$, egyébként div.

$\sum_{n=1}^{\infty}\frac{1}{n^\alpha}$ konv, ha $\alpha>1$, egyébként div.

($\alpha<0$ $\sum n^{|\alpha|}\to\infty$

$\sum^\infty \frac{1}{n^{1{,}000\dots01}}$ konv $s\neq s_n$

$\sum^\infty \frac{1}{n\cdot\log_2 n}$ div (int. krit.))

a) $\sum\frac{1}{n}$ jellegű sor ⇒ minoráns $(\to\infty)$

$a_n=\frac{2n+3}{n^2+8}>\frac{2n}{n^2+8n^2}=\frac{2n}{9n^2}=\frac{2}{9}\cdot\frac{1}{n}=d_n$

$d_n=\frac{2}{9}\sum\frac{1}{n}$ div ($\alpha=1$) ⇒ minoráns krit.: a_n div.

b) $b_n = \left(\frac{2n+3}{n^2+8}\right)^2 \le \left(\frac{2n+3n}{n^2}\right)^2 = \left(\frac{5}{n}\right)^2 = 25\cdot\frac{1}{n^2}$

$25\cdot\sum_1 \frac{1}{n^2}$ konv $(\alpha=2>1) \underset{\text{maj. kr.}}{\Longrightarrow} \sum_1^\infty b_n$ konv

c) $\left(\frac{2n+3}{n^2+8}\right)^{\frac{1}{n}}$

$\underbrace{\underbrace{\sqrt[n]{\frac{1}{3}}}_{\to 1}\cdot\underbrace{\frac{1}{\sqrt[n]{n^2}}}_{\to 1}}_{\to 1} = \sqrt[n]{\frac{3}{n^2+8n^2}} \le c_n = \sqrt[n]{\frac{2n+3}{n^2+8}} \le \sqrt[n]{\frac{2n+3n}{8}} = \underbrace{\underbrace{\sqrt[n]{\frac{5}{8}}}_{\to 1}\cdot\underbrace{\sqrt[n]{n}}_{\to 1}}_{\to 1}$

$c_n \to 1 \neq 0 \Rightarrow \sum_1^\infty c_n$ div, mert nem teljesül a konv szükséges feltétele

(12) Konv-e?

a) $\sum_1^\infty \frac{n^2-3\sqrt{n}}{2n^3-n^2+5n+7}$ b) $\sum_1^\infty \frac{n^2-3\sqrt{n}}{n^4-n^2+5}$

a) $a_n \ge \frac{n^2-\frac{n^2}{2}}{2n^3+0+5n^3} = \frac{n^2}{14n^3} = \frac{1}{14}\cdot\frac{1}{n}$

$\frac{1}{14}\cdot\sum_1 \frac{1}{n}$ div $\underset{\text{maj. kr.}}{\Longrightarrow} \sum_1^\infty a_n$ ~~konv~~ div

b) $b_n \le \frac{n^2-0}{n^4-\frac{n^4}{2}+0} = 2\cdot\frac{1}{n^2}$ $2\cdot\sum_1^\infty \frac{1}{n^2}$ konv $(\alpha=2>1) \underset{\text{maj. kr.}}{\Longrightarrow} \sum_1 b_n$ konv.

(12) a) $\sum_1^\infty \left(\frac{n+1}{4n+2}\right)^n$ b) $\sum_1^\infty \left(\frac{3n+1}{4n+2}\right)^n$ c) $\sum_1^\infty \left(\frac{4n+1}{4n+2}\right)^n$ d) $\sum_1^\infty \left(\frac{4n+1}{3n+2}\right)^n$

a) $a_n < \left(\frac{n+n}{4n+0}\right)^n = \left(\frac{1}{2}\right)^n$ $\sum_1 \left(\frac{1}{2}\right)^n$ konv. geom. sor $\left(q=\frac{1}{2}<1\right) \Rightarrow \sum_1 a_n$ konv

b) $b_n < \left(\frac{3n+\frac{n}{2}}{4n+0}\right)^n = \left(\frac{7n}{8n}\right)^n = \left(\frac{7}{8}\right)^n$ $\sum_1 \left(\frac{7}{8}\right)^n$ konv. geom. sor $\left(-1<0<q=\frac{7}{8}<1\right)$

$\Downarrow$ $\sum_1^\infty b_n$ konv.

c) $c_n = \frac{\left(1+\frac{1/4}{n}\right)^n}{\left(1+\frac{1/2}{n}\right)^n} \to \frac{e^{1/4}}{e^{1/2}} = e^{-1/2} \neq 0 \Rightarrow \sum_1 c_n$ div (szükséges felt. nem teljesül)

d) $d_n = \left(\frac{4n+1}{3n+2}\right)^n \underset{n\ge 4}{\ge} \left(\frac{4n}{3n+\frac{n}{2}}\right)^n = \left(\frac{4}{3,5}\right)^n \to \infty \Rightarrow \sum_1 d_n$ div (nem teljesül a szüks. felt.)

vagy $d_n \ge \left(\frac{4}{3,5}\right)^n$ div. geom. sor $\left(q=\frac{4}{3,5}>1\right) \underset{\text{min. kr.}}{\Longrightarrow} \sum_1^\infty d_n$ div.

Analízis elmélet

2007.10.01.
Kányan

(Pl) a) $\sum^{\infty} \frac{2^n+5^n}{6^{n+1}+3^{n-1}}$ b.) $\sum_1^{\infty} \frac{3^{2n}+1}{2^{3n}+5}$

a) $\sum^{\infty} \frac{2^n+5^n}{6\cdot 6^n+\frac{1}{3}\cdot 3^n}$ $a_n \le \frac{5^n+5^n}{6^n} = 2\left(\frac{5}{6}\right)^n$ $2\sum\left(\frac{5}{6}\right)^n$ konv. geom. sor ($q=\frac{5}{6}<1$) így $\sum^{\infty} a_n$ konv.

b) $\sum^{\infty} \frac{9^n+1}{8^n+5}$ $b_n \ge \frac{9^n}{8^n+5\cdot 8^n} = \frac{1}{6}\cdot\left(\frac{9}{8}\right)^n$

$\frac{1}{6}\sum^{\infty}\left(\frac{9}{8}\right)^n$ div. geom. sor $\left(q=\frac{9}{8}>1\right) \underset{\text{min. kr.}}{\Rightarrow} \sum^{\infty} b_n$ div.

(Pl) a) $\sum^{\infty} \frac{2^n+5^n}{6^{n+1}}$ b.) $\sum^{\infty} \frac{5^n}{6^{n+1}+3^{n-1}}$ Mut. meg, hogy konv. Adja meg a sor összegét vagy becsülje meg s-et.

a) $\sum_{n=1}^{\infty} \frac{1}{6}\cdot\left(\left(\frac{2}{6}\right)^n+\left(\frac{5}{6}\right)^n\right) = \frac{1}{6}\cdot\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n + \frac{1}{6}\cdot\sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^n = \frac{1}{6}\cdot\frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{1}{6}\cdot\frac{\frac{5}{6}}{1-\frac{5}{6}}$

b.) $\sum_1^{\infty} \frac{5^n}{6\, 6^n+\frac{1}{3}\cdot 3^n}$

$b_n \le \frac{5^n}{6^n} = \left(\frac{5}{6}\right)^n$ $\sum^{\infty}\left(\frac{5}{6}\right)^n$ konv. geom. sor $\Rightarrow \sum_1^{\infty} b_n$ k

Hibaszámítás ($s \approx s_n$)

Pozitív tagú soroknál

$\underset{b_n>0}{0 <} H = \sum_{k=n+1}^{\infty} b_k < \sum_{k=n+1}^{\infty} c_k = \ldots$

$s \approx s_{1000}$

$0 < H = \sum_{n=1001}^{\infty} \frac{5^n}{6\cdot 6^n+\frac{1}{3}\cdot 3^n} \le \sum_{n=1001}^{\infty} \frac{5^n}{6\cdot 6^n} = \frac{1}{6}\cdot\sum_{n=1001}^{\infty}\left(\frac{5}{6}\right)^n = \frac{1}{6}\cdot\frac{\left(\frac{5}{6}\right)^{1001}}{1-\frac{5}{6}}$

(Pl) Abszolút vagy felt. konvergens-e?

a) $\sum^{\infty} (-1)^{n+1} \frac{n^2+4}{n^2+n}$ b.) $\sum_1^{\infty} (-1)^{n+1} \frac{n^2+4}{n^5+7}$ c) $\sum_1^{\infty} (-1)^{n+1} \frac{n+4}{n^2+n}$

c). $\sum (-1)^{n+1} \cdot \underbrace{\frac{n+4}{n^2+n}}_{:=c_n}$ 1.) Absz. konv. e? $\sum^{\infty} \frac{n+4}{n^2+n}$ $c_n \ge \frac{n}{n^2+n^2} = \frac{1}{2}\cdot\frac{1}{n}$

$\frac{1}{2}\cdot\sum^{\infty}\frac{1}{n}$ div $\Rightarrow \sum^{\infty} c_n$ div, tehát nem absz. konv

Leibniz-sor-e?

$c_n \searrow 0$

$$c_n = \frac{n+4}{n^2+n} = \underbrace{\frac{n}{n^2}}_{=\frac{1}{n}} \cdot \underbrace{\frac{1+\frac{4}{n}}{1+\frac{1}{n}}}_{\to 1} \to 0$$

Mon. csökk.-e? $c_{n+1} < c_n$

$$\frac{n+5}{(n+1)^2+(n+1)} \overset{?}{<} \frac{n+4}{n^2+n}$$

$$(n+5)(n^2+n) \overset{?}{<} (n+4)(n^2+3n+2)$$

$$n^3+5n^2+n^2+5n \overset{?}{<} n^3+3n^2+2n+4n^2+12$$

$$0 \overset{?}{<} n^2+9n+8 \quad ✓$$

$\Downarrow$

Leibniz-sor, tehát a sor felt. konv.

Analízis elmélet

2007.10.10
Kónya

(PE) Abszolút vagy felt. konvergens-e az alábbi sor?

1) $\sum^{\infty} \overbrace{(-1)^n \frac{1}{\sqrt[3]{n}}}^{a_n}$

Absz: $\sum^{\infty} |a_n| = \sum^{\infty} \frac{1}{\sqrt[3]{n}} = \sum^{\infty} \frac{1}{n^{1/3}}$ ~~konv~~ $\alpha = \frac{1}{3} < 1$ div $\Rightarrow$ nem absz konv

$|a_n| = \frac{1}{\sqrt[3]{n}} \searrow 0$ ✓ $\Rightarrow$ ~~felt. konv~~ Leibniz-sor, tehát konv

Tehát felt. konv.

2) $\sum^{\infty} \underbrace{(-1)^n \frac{1}{\sqrt[3]{n} + (\sqrt[3]{2})^n}}_{a_n}$

$|a_n| = \frac{1}{\sqrt[3]{n} + (\sqrt[3]{2})^n} \le \frac{1}{(\sqrt[3]{2})^n}$ $\qquad \sum^{\infty} \frac{1}{(\sqrt[3]{2})^n}$ konv. geom. sor $\left(0 < q = \frac{1}{\sqrt[3]{2}} < 1\right)$

majoráns kr.

$\sum^{\infty} (a_n)$ konv, tehát absz. konv

3.) a) $\sum^{\infty} (-1)^{n+1} \frac{n^2+4}{n^2+n}$ $\qquad$ b) $\sum^{\infty} (-1)^{n+1} \frac{n^2+4}{n^5+7}$ $\qquad$ c) $\sum^{\infty} (-1)^{n+1} \frac{n+4}{n^2+n}$

a) $\sum a_n$

$|a_n| \to 1 \neq 0$ $\quad$ a sor divergens, mert nem teljesül a szüks. felt.

b) $\sum b_n$

$|b_n| = \frac{n^2+4}{n^5+7} < \frac{n^2+4n^2}{n^5} = 5 \cdot \frac{1}{n^3}$

$5 \cdot \sum^{\infty} \frac{1}{n^3}$ konv $(\alpha = 3 > 1)$ $\Rightarrow \sum^{\infty} |b_n|$ konv, tehát a sor absz. konv

c.) $\sum^{\infty} c_n$

$|c_n| = \frac{n+4}{n^2+n} \ge \frac{n+0}{n^2+n^2} = \frac{1}{n} \cdot \frac{1}{2}$

$\frac{1}{2} \sum^{\infty} \frac{1}{n}$ div $\Rightarrow$ nem absz. konv.

$\sum |c_n|$ div.

Leibniz-sor-e? $|c_n| = \frac{n+4}{n^2+n} = \frac{n}{n^2} \cdot \frac{1 + \frac{4}{n}}{1 + \frac{1}{n^2}} \to \frac{1}{n} \cdot \frac{1}{1} \to 0$

$|c_n| \searrow^? : |c_n| \geq^? |c_{n+1}|$

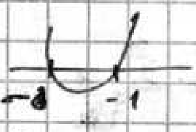
$$\frac{n+4}{n^2+n} \geq^? \frac{n+1+4}{(n+1)^2+n+1}$$

$$\vdots$$

$$(n+4)(n^2+3n+2) \geq^? (n^2+n)(n+5)$$

$$n^3+7n^2+14n+8 \geq^? n^3+6n^2+5n$$

$$n^2+9n+8 \geq^? 0 \quad \checkmark$$



Tehát a $\sum_1^\infty c_n$ sor

~~a sorozat~~ $|c_n| \searrow 0 \Rightarrow$ Leibniz-sor $\Rightarrow$ felt. konv.

4.) a) $\sum_1^\infty \overbrace{(-1)^n \frac{3^n}{n^2\cdot 3^n+2}}^{a_n}$ b) $\sum_1^\infty (-1)^n \frac{3^n}{n\cdot 3^n+2}$

Absz?: $|a_n| = \frac{3^n}{n^2\cdot 3^n+2} = \underbrace{\frac{3^n}{3^n}}_{=1}\cdot\frac{1}{n^2+\frac{2}{3^n}} \to \frac{1}{n^2}$ vagy majoráns krit.

$|b_n|$ nem absz. konv (minoráns krit.)

Leibniz sor-e? $b_n \to 0$; $\searrow$

(Pl) Konv-e az alábbi sor? Ha a sor konv, akkor becsülés az $s \approx s_{100}$ becslés hibáját

Ha pozitív tagú: Pl: $0 < H < 10^{-5}$

Ha nem: $|H| < 10^{-5}$

1.) a) $\sum_1^\infty \frac{2^n+5^n}{6^{n+1}}$ b.) $\sum_1^\infty \frac{5^n}{6^{n+1}+3^{n-1}}$

2.) $\sum_1^\infty \frac{n\cdot 3^n}{(n+1)^2\cdot 2^{3n}}$ 4.) $\sum_1 (-1)^{n+1}\frac{n}{5^{n+2}}$ 6.) $\sum_1 \frac{(-1)^n}{\sqrt[n]{\lg 2}}$

3.) $\sum_1^\infty \frac{2^n+5^n}{2^{2n+1}+n^2+n}$ 5.) $\sum_1 \frac{(-1)^n}{\lg 2^n}$

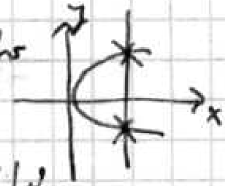
Analízis elmélet

2007.10.10
Kónya

Valós egyváltozós függvények

fv: $f: D_f \to R_f$ $\quad D_f, R_f \subseteq \mathbb{R}$

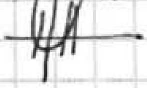
$x = y^2$ nem fv



egyértékű reláció

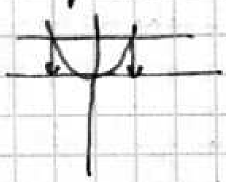
$y = f(x)$

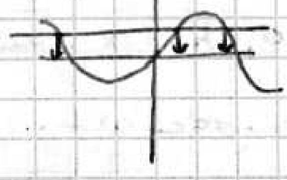
$y = x^2$ fv



Egyrétű $\qquad$ nem egyrétű

$y = x$ $\qquad$ $y = x^2$ $\qquad$ $y = \sin x$





Egyértékű és egyrétű ≡ k.e.é.l. = 1-1 értelmű.

$f: \mathbb{R} \to \mathbb{R}$

f korlátos: $\exists K : |f(x)| \le K$

f monoton nő: $\forall x_1 < x_2 \quad (x_1, x_2 \in D_f): f(x_1) \le f(x_2)$

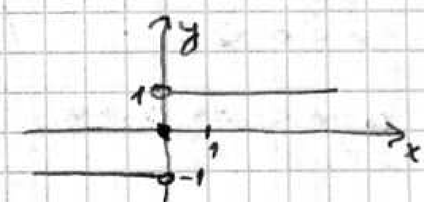
f periodikus: $\exists p > 0 : f(x+p) = f(x) \qquad x \in \mathbb{R}$

paritás: ps: $f(-x) = f(x)$ $\qquad$ ÉT szimm. az origóra

pt: $f(-x) = -f(x)$ $\qquad$ — " —

Néhány fv:

Előjel fv: $\quad \mathrm{sg}\, x = \mathrm{sign}\, x = \begin{cases} 1, & \text{ha } x > 0 \\ 0, & \text{ha } x = 0 \\ -1, & \text{ha } x < 0 \end{cases}$

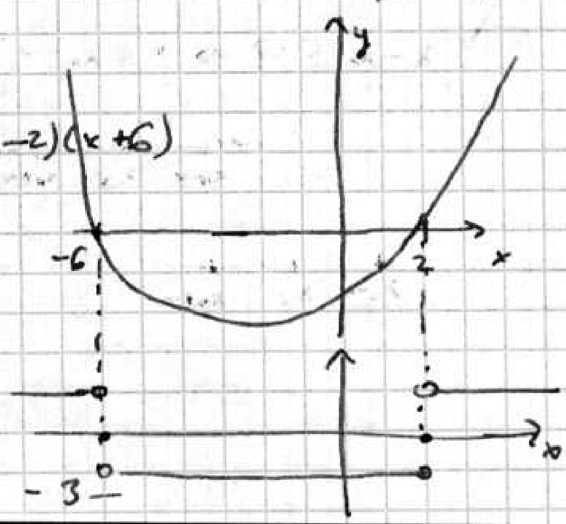


$f(0+0) = 1$

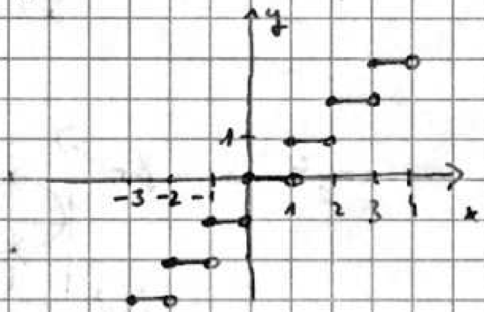
$f(0-0) = -1$

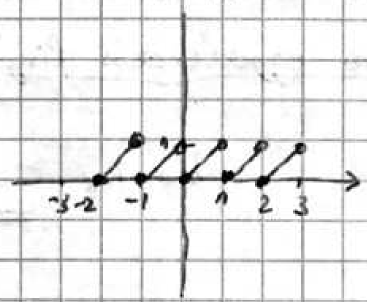
(Fe). Ábrázoljuk:

$\mathrm{sg}(x^2 + 4x - 12) = \mathrm{sg}\,(x-2)(x+6)$



Egészrész fv $[x]$: x-nél nem nagyobb legnagyobb egész:

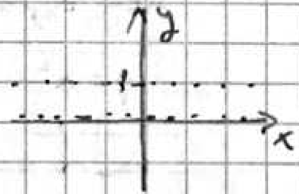




$(p=1)$

Törtrész fv $\{x\} = x-[x]$

Dirichlet fv:
$$f(x)=\begin{cases}1, & \text{ha } x \text{ rac}\\ 0, & \text{ha } x \text{ irrac}\end{cases}$$



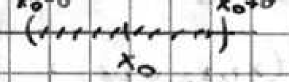
periódus: bármely poz. rac. szám

——— o ———

(D) x_0 a $H\subset\mathbb{R}$ torlódási pont, ha $\forall$ környezete H végtelen sok elemét tartalmazza

Jelölések:

$K_{x_0,\delta} = K(x_0,\delta)$



$(x_0-\delta;\ x_0+\delta)$

$\dot{K}_{x_0,\delta}$ $\quad (x_0-\delta;\ x_0+\delta)\setminus\{x_0\}$

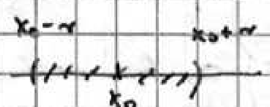
pontozott környezete

Ha δ-nak nincs szerepe: K_{x_0}

$|x-x_0|<r$

$-r<x-x_0<r \quad /+x_0$

$x_0-r<x<x_0+r$



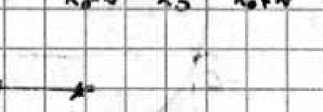
$|x+1|<2$

$|x-(-1)|<2$

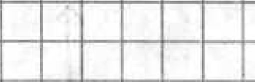
a (-1) 2 sugarú környezete

$|x-x_0|>r \to x-x_0>r \Rightarrow x>x_0+r$

$\qquad\quad\ \ \to x-x_0<-r \Rightarrow x<x_0-r$



$|x-x_0|=r$



Analízis elmélet

2007-10-11.
Tkónya

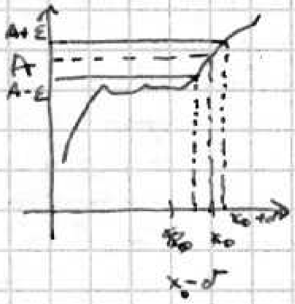
Ⓓ $\lim_{x\to x_0} f(x) = A \in \mathbb{R}$

$\forall\ \varepsilon > 0$-hoz $\exists\ \delta(\varepsilon) > 0$

$|f(x) - A| < \varepsilon$, ha

$0 < |x - x_0| < \delta(\varepsilon)$

$A - \varepsilon < f(x) < A + \varepsilon$



Ⓟⓛ Biz. be., hogy $\lim_{x\to 3} (2x+5) = 11$

$$|f(x) - A| \underset{x_0=3}{=} |2x+5-11| = |2x-6| = 2|x-3| < \varepsilon \quad \left\{ \underset{x\neq 3}{(0<)} |x-3| < \underset{\delta(\varepsilon)}{(\ldots)} \right.$$

$$|x-3| < \frac{\varepsilon}{2}$$

$$\delta(\varepsilon) = \frac{\varepsilon}{2}$$

Ⓟⓛ $\lim_{x\to -3} \left(\frac{x^2-9}{x+3} - 5x + 5 \right) = 14$

$$f_1(x) = \frac{x^2-9}{x+3} \neq x-3 = f_2(x)$$

$$|f(x) - A| = \left| \frac{x^2-9}{x+3} - 5x + 5 - 14 \right| \underset{x\neq -3}{=} |x-3-5x+5-14| = \underset{-4(x+3)}{|-4x-12|} = 4|x+3| < \varepsilon$$

beépítem

$$\underset{x\neq -3}{(0<)} |x-(-3)| = |x+3| < (\ldots)$$

$$|x+3| < \frac{\varepsilon}{4} = \delta$$

Ⓟⓛ $\lim_{x\to 3} \sqrt{2x+3} = 3$

$$|f(x) - A| = |\sqrt{2x+3} - 3| = \left| (\sqrt{2x+3} - 3) \cdot \frac{\sqrt{2x+3}+3}{\sqrt{2x+3}+3} \right| = \left| \frac{2x+3-9}{\sqrt{2x+3}+3} \right| = \left| \frac{2x-6}{\sqrt{2x+3}+3} \right| = \frac{2|x-3|}{\sqrt{2x+3}+3}$$

$$|x-3| < (\ldots) \quad \left\{ \quad < \frac{2|x-3|}{0+3} < \varepsilon \Rightarrow |x-3| < \frac{3\varepsilon}{2} = \delta(\varepsilon) \right.$$

Analízis elmélet

2007. 10. 18.
Kónya

(Pl) HF: $\lim_{x\to\pm\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x+2}\right)=?$

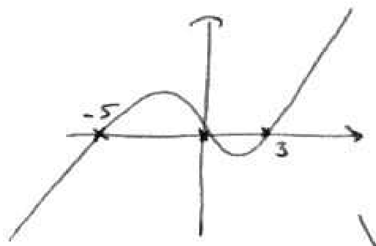
(Pl) Vázlatosan ábrázoljuk:

$f_1(x)=(x+5)\,x\,(x-3)$

$f_2(x)=(1-x)(x-2)^2$

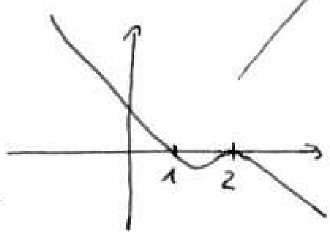
$f_1(x)=x^3+\dots=x^3\big(\underbrace{1+\dots}_{\to 1}\big)$

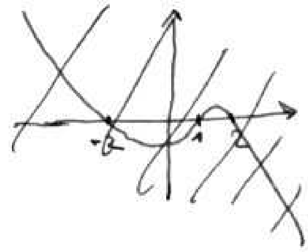
$\lim_{x\to\pm\infty} f_1(x)=\pm\infty$



$f_2(x)=-x^3+\dots$

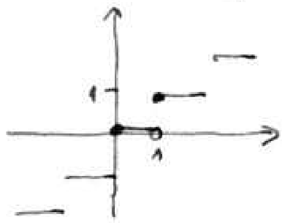
$x\approx 2 \quad f_2(x)\approx-(x-2)^2$





(Pl) a) $\lim_{x\to 0}\frac{[x]}{x}=?$ b) $\lim_{x\to 0}\frac{\{x\}}{x}=?$

a)

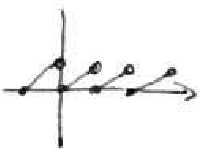


$\lim_{x\to 0+0}\frac{[x]}{x}\underset{0<x<1}{=}\frac{0}{x}=0$ nem határozatlan, nulla kis szám

$\lim_{x\to 0-0}\frac{[x]}{x}\underset{-1<x<0}{=}\lim_{x\to 0-0}\frac{-1}{\underset{\to 0}{x}}=+\infty$

$\lim_{x\to 0}\frac{[x]}{x}$ nem létezik
másodfajú szakadás

b)



$\lim_{x\to 0+0}\frac{\{x\}}{x}\underset{0<x<1}{=}\lim_{x\to 0+0}\frac{x}{x}=1$

$\lim_{x\to 0-0}\frac{\{x\}}{x}\underset{-1<x<0}{=}\lim_{x\to 0-0}\frac{\overset{\to 0}{x}+1}{\underset{\to -0}{x}}=-\infty$

$\lim \frac{\{x\}}{x}$ nem létezik
másodfajú szakadás

vagy $\lim_{x\to 0-0}\frac{\overset{\to 1}{\{x\}}}{\underset{\to -0}{x}}=-\infty$

vagy $\{x\}=x-[x]$ $\qquad \frac{\{x\}}{x}=\frac{x-[x]}{x}=1-\underbrace{\frac{[x]}{x}}_{a.)}$

(T) $x,\ x^2,\ x^n,\ P_n(x)$ mindenütt folytonos.

$f_1(x)=x$

(D) $\forall\,\varepsilon>0$-hoz $\exists\,\delta(\varepsilon)$ $\quad |f(x)-f(x_0)|<\varepsilon$, ha $|x-x_0|<\delta(\varepsilon)$

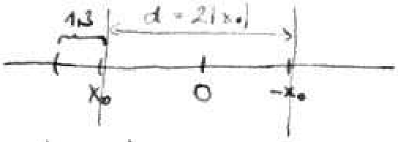
$|f_1(x)-f_1(x_0)|=|x-x_0|<\varepsilon \Rightarrow \delta(\varepsilon)=\varepsilon$ $\qquad |x-x_0|<\dots$ kell

HF: $f_1(x)=3x+5$ mindenütt folytonos.

$f_2(x) = x^2$

$$|f_2(x) - f_2(x_0)| = |x^2 - x_0^2| = |x - x_0| \cdot \underbrace{|x+x_0|}_{\text{ennek a helyére } r\text{-t}} < |x-x_0| \cdot (2|x_0| + 1{,}3) < \varepsilon \qquad |x-x_0| < \ldots = \delta(\varepsilon)$$

$$\delta(\varepsilon) = \min\left\{\frac{\varepsilon}{2|x_0|+1{,}3},\ 1{,}3\right\}$$



$r := 2|x_0| + 1{,}3$ ← ez mindegy, hogy mekkora, csak $r > 0$

$|x-a| < r$ $a-r \quad a \quad a+r$

$|x + x_0| = |x - (-x_0)| < r$

Másik mo: $f_2(x) = \underset{\text{folyt}}{f_1(x)} \cdot \underset{\text{folyt}}{f_1(x)} \Rightarrow f_2$ folyt.

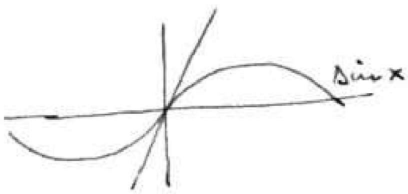
$f_3(x) = x^n$ teljes indukcióval.

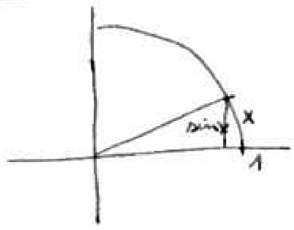
$$\left(|f_3(x) - f_3(x_0)| = |x^n - x_0^n| = |x - x_0| \cdot |x^{n-1} + x^{n-2}x_0 + \ldots| \right)$$

$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

folytonos fv-k ~~és~~ összetétele folytonos

— o —





$\sin x < x \quad x \in \left(0; \frac{\pi}{2}\right) \ldots$

$|\sin x| \le |x| \quad \forall\ x$-re

Pl. Konv-e? $\sum_1^\infty \frac{\sin\frac{1}{n}}{n}$

$$0 < a_n \le \frac{\sin\frac{1}{n}}{n} \le \frac{\frac{1}{n}}{n} = \frac{1}{n^2} \qquad \sum_1^\infty \frac{1}{n^2} \text{ konv } (\alpha = 2 > 1)$$

$\Downarrow$

$\sum_1 a_n$ konv

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

Pl. $\lim_{x\to 0} \frac{\sin 9x}{7x} = \lim_{x\to 0} \frac{\sin 9x}{9x} \cdot \frac{9}{7} \overset{u:=9x}{=} \lim_{u\to 0} \frac{\sin u}{u} \cdot \frac{9}{7} = \frac{9}{7}$

Pl. $\lim_{x\to 0} \frac{\sin 2x}{\sin 9x} = \lim_{x\to 0} \frac{\frac{\sin 2x}{2x}}{\frac{\sin 9x}{9x}} \cdot \frac{2\cancel{x}}{9\cancel{x}} \to \frac{2}{9}$ ($\frac{\sin 2x}{2x} \to 1$, $\frac{\sin 9x}{9x} \to 1$)

$\left(\lim_{x\to 0} \frac{\arcsin 5x^2}{\operatorname{arctg} 3x^2} \overset{L'H}{=} \lim \frac{\frac{1}{\sqrt{1-(5x^2)^2}} \cdot 10\cancel{x}}{\frac{1}{1+(3x^2)^2} \cdot 6\cancel{x}} = \frac{10}{6} \right.$

Pl. $\lim_{x\to 0} \frac{\operatorname{tg} 3x^2}{\sin 5x^2} \overset{0/0}{=} \lim_{x\to 0} \frac{\sin 3x^2}{3x^2} \cdot \frac{1}{\cos 3x^2} \cdot \frac{3\cancel{x^2}}{5\cancel{x^2}} \cdot \frac{5x^2}{\sin 5x^2} \to \frac{3}{5}$ ($\to 1$, $\to 1$, $= 1$, $\to 1$)

Pl. HF. $\lim_{x\to 0} \frac{\sin 3x^4}{\operatorname{tg}^2 4x \cdot \sin 5x^2} =$

Analízis elmélet

2007. 10. 18
Kónya

Pl $\lim_{x\to 2} \frac{\sin\sqrt{x^2-4x+4}}{x^2-4} = \lim_{2} \frac{\sin\sqrt{(x-2)^2}}{(x-2)(x+2)} = \lim_{2} \frac{\sin|x-2|}{(x-2)(x+2)}$

$\lim_{2+0} \underbrace{\frac{\sin(x-2)}{x-2}}_{\to 1} \cdot \underbrace{\frac{1}{x+2}}_{\to \frac{1}{4}} \Rightarrow \frac{1}{4}$

$\lim_{2-0} -\underbrace{\frac{\sin(-(x-2))}{-(x-2)}}_{\to 1} \cdot \underbrace{\frac{1}{x+2}}_{\to \frac{1}{4}} \to -\frac{1}{4}$

$\Bigg\}$ $\lim_{x\to 2} \dots \nexists$

véges ugrás (elsőfajú szakadás)

Pl $\lim_{x\to 0} \frac{3x+\sin 2x}{x+\sin 7x} = \lim_{x\to 0} \frac{3+\overbrace{\frac{\sin 2x}{x\cdot 2}}^{\to 1}\cdot 2}{1+\underbrace{\frac{\sin 7x}{x\cdot 7}}_{\to 1}\cdot 7} \Rightarrow \frac{3+2}{1+7}$

Pl $\lim_{x\to\infty} \frac{3x+\sin 2x}{x+\sin 7x} = \lim_{x\to\infty} \frac{3+\frac{\sin 2x}{x}}{1+\frac{\sin 7x}{x}} = \frac{3+0}{1+0}$

$\underbrace{-\frac{1}{x}}_{\to 0} < \underbrace{\frac{\sin 2x}{x}}_{\to 0} < \underbrace{\frac{1}{x}}_{\to 0}$ vagy $\underbrace{\frac{1}{x}}_{\to 0} \cdot \underbrace{\sin 2x}_{\text{korl.}} \Rightarrow 0$

$\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \to 1$ nev. hat. é.

$\lim_{x\to 0} \underbrace{x}_{\to 0} \cdot \underbrace{\sin\frac{1}{x}}_{\text{korl.}} = 0$ nem ordít, mert nevezetes határérték.

Pl $\boxed{\lim_{x\to 0} \frac{\cos x - 1}{x}}$

$\cos^2 x + \sin^2 x = 1$
$\cos^2 x - \sin^2 x = \cos 2x$

$\boxed{\begin{array}{l} (+)\ \cos^2 x = \frac{1+\cos 2x}{2} \\ (-)\ \sin^2 x = \frac{1-\cos 2x}{2} \end{array}}$

$1-\cos 2u = 2\sin^2 u$

$\lim_{0} \frac{\cos x - 1}{x} = \lim_{0} \frac{-2\sin^2\frac{x}{2}}{x} = \lim_{0} -\overbrace{\frac{\sin\frac{x}{2}}{\frac{x}{2}}}^{\to 1} \cdot \underbrace{\sin\frac{x}{2}}_{\to 0} = 0$

Pl HF: $\lim_{x\to 0} \frac{1-\cos 5x}{7x^2}$

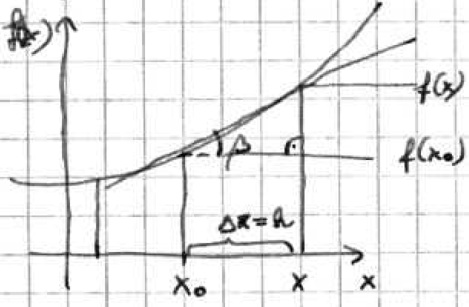
Analízis elmélet

2007.10.24.
Kónya

Differenciálszámítás

derivált = differenciálhányados

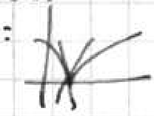
(a fv. változási sebessége) az érintő meredeksége



$$tg\,\beta = \frac{f(x)-f(x_0)}{x-x_0} = \frac{\Delta f}{\Delta x}$$:

differenciahányados / különbségi hányados

(ha a jobboldali és a baloldali érintő ugyanaz.
Betg:



)

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$ differenciálhányados / derivált

Pl. $f(x) = x^2$ $\quad f'(5) = ?$ $\quad f'(x_0) = ?$ $\quad f'(x) = ?$

$$\left(f'(x) = \frac{d}{dx} f = \frac{df}{dx}\right)$$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Ált. nem. f(x)-f(x₀) ala...

$f(x)-f(x_0) = \sin x - \sin$...
helyett
$f(x_0+h) - f(x_0) = \sin(x_0+h)-$...

$$f'(5) = \lim_{h \to 0} \frac{f(5+h)-f(5)}{h} = \lim_{h \to 0} \frac{(5+h)^2 - 5}{h} = \lim_{h \to 0} \frac{10h + h^2}{h} = 10$$

$$\frac{h}{h}(10+h)$$

Anal. elmélet

2007. 10. 31.
Kónya

Pl $\left(\frac{1}{\cos^2 3x+5}\right)' = ?$

1. mo. $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2} \to \left(\frac{1}{\cos^2 3x+5}\right)' = \frac{2\cos 3x \cdot (-\sin 3x) \cdot 3 + 0}{(\cos^2 3x+5)^2}$

2. mo $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2} \to \left(\frac{1}{\cos^2 3x+5}\right)' = \frac{0 - 1(\cos^2 3x+5)'}{(\cos^2 3x+5)^2}$

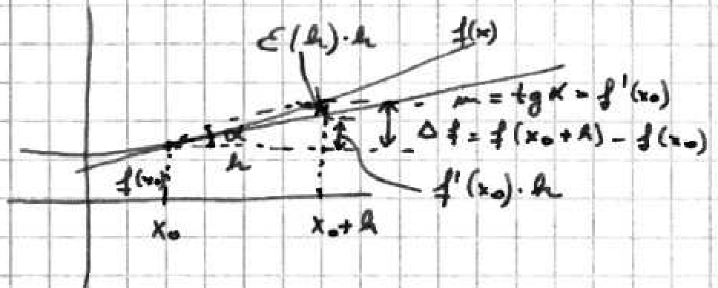
3. mo $\left(\frac{1}{\cos^2 3x+5}\right)' = \left((\cos^2 3x+5)^{-1}\right)' = -1 \cdot (\cos^2 3x+5)^{-2} \cdot (\cos^2 3x+5)' = \ldots$

$\boxed{(f \cdot g \cdot h)'} = (f \cdot g)' \cdot h + (f \cdot g) \cdot h' = \boxed{f' \cdot g h + f \cdot g' h + f \cdot g \cdot h'}$

Differenciál

T deriválhatóság $\Leftrightarrow$ $\Delta f = f(x_0+h) - f(x_0) = \underbrace{f'(x_0)\, h}_{\text{fő rész}} + \underset{\to 0}{\varepsilon(h)} \cdot h$

fő rész
differenciál



differenciál

$d f(x_0, h) := f'(x_0) \cdot h$

$d f(x_0, \Delta x) = f'(x_0) \cdot \Delta x$

(∀ deriváltat tudunk így is definiálni, de van olyan, amit csak így lehet).

$\Delta f \approx d f$

$f(\underbrace{x_0+h}_{:=x}) - f(x_0) \approx f'(x_0) \cdot h \Rightarrow \boxed{f(x) \approx f(x_0) + f'(x_0) \cdot (x-x_0)}$

egyenes egyenlete: $y = y_0 - m(x-x_0)$

$f(x) := \sin x$ $\qquad f'(x) = \cos x$

$d f(x,h) = f'(x) \cdot h = \cos x \cdot h$

$d f = \cos x \cdot h$

$d \sin x = \cos x \cdot h$

$d x = 1 \cdot h$ $\Rightarrow$ h leccserélhető az x szimbólumra.

$(f(x) = x)$

$d \sin x = \cos x \, dx$

$(\sin x)' = \frac{d}{dx} \cdot \sin x = \frac{d \sin x}{dx} = \cos x$

$d \sin x = \cos x \cdot dx$

formálisan is kijön.

Magasabbrendű deriváltak

$\exists\ f'(x)$ K_{x_0}-ban: $f''(x_0) = \lim_{h\to 0} f'(x_0+h) - f'(x_0)$

Jelölés: $f''(x)$; $\frac{d}{dx}\cdot\frac{d}{dx} f = \frac{d^2 f}{dx^2}$

← arab számmal zárójelben.

$f'''(x)$, $f^{IV}(x)$, $f^{V}(x)$ vagy $f^{(5)}(x) = \frac{d^5 f}{dx^5}$, $f^{(n)}(x)$

Pl $f(x) = |2x-1|\cdot\sin(2x-1)$ $|x|$ nem deriválható a 0 helyen.

$f'(x) = ?$

$g(x) := (2x-1)\cdot\sin(2x-1)$

$g'(x) = (2x-1)'\cdot\sin(2x-1) + (2x-1)\cdot(\sin(2x-1))' = 2\cdot\sin(2x-1) + (2x-1)\cdot\cos(2x-1)\cdot 2$

$$f(x) = \begin{cases} (2x-1)\cdot\sin(2x-1) = g(x), & \text{ha } x \geq \frac{1}{2} \\ -(2x-1)\sin(2x-1) = -g(x), & \text{ha } x < \frac{1}{2} \end{cases}$$

← folytonosan csatlakozik, csak nem biztos, hogy deriválható

$$f'(x) = \begin{cases} g'(x), & \text{ha } x > \frac{1}{2} \\ -g'(x), & \text{ha } x < \frac{1}{2} \\ 0, & \text{ha } x = \frac{1}{2} \end{cases}$$

(a csatlakozási pontokon egyszerűbb a def-vel)

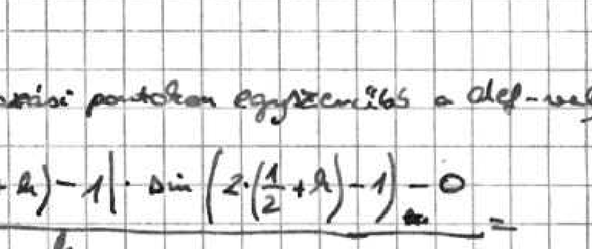
$$f'\left(\frac{1}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{1}{2}+h\right) - f\left(\frac{1}{2}\right)}{h} = \lim_{h\to 0} \frac{\left|2\cdot\left(\frac{1}{2}+h\right)-1\right|\cdot\sin\left(2\cdot\left(\frac{1}{2}+h\right)-1\right) - 0}{h} =$$

folytonos, tehát $\frac{0}{0}$

$$= \lim_{h\to 0} \underbrace{2|h|}_{\to 0} \cdot \underbrace{\frac{\sin 2h}{h\cdot 2}}_{\to 1} \cdot 2 = 0$$

Pl $f(x) = \underbrace{\sqrt[5]{x^4}}_{x^{\frac{4}{5}}} \cdot \sin \underbrace{\sqrt[5]{x}}_{x^{\frac{1}{5}}}$ $f'(x) = ?$ ← mindenütt folytonos

$f'(x) = \frac{4}{5}\cdot \underbrace{x^{-\frac{1}{5}}}_{\frac{1}{\sqrt[5]{x}}} \cdot \sin\sqrt[5]{x} + \sqrt[5]{x^4}\cdot\cos\sqrt[5]{x}\cdot\frac{1}{5}\cdot x^{-\frac{4}{5}}$



$\sqrt[3]{x}$ nem deriválható a 0-ban!

$\sqrt[n]{x}$ NEM DERIVÁLHATÓ 0-ban!

$(\sqrt[3]{x})' = \frac{1}{3}\cdot x^{-\frac{2}{3}}$ $x \neq 0$

$\nexists$, ha $x = 0$

$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \frac{0}{0}\ \frac{\sqrt[5]{h^4}\sin\sqrt[5]{h} - 0}{h} =$

$h = \sqrt[5]{h^5} = \sqrt[5]{h^4 h} = \sqrt[5]{h^4}\cdot\sqrt[5]{h}$

$= \lim_{h\to 0} \frac{\sqrt[5]{h^4}}{\sqrt[5]{h^4}} \cdot \frac{\sin\sqrt[5]{h}}{\sqrt[5]{h}} = 1$

Anal elmélet

2007.10.31.
Kbnya

Pl

$$f(x)=\begin{cases} x^2\cdot\sin\frac{1}{x}, & \text{ha } x\neq 0\\ 0, & \text{ha } x=0\end{cases} \qquad f'(x)=? \qquad f''(x)=?$$

$$\lim_{x\to 0} \underset{\downarrow\atop 0}{x^2}\cdot\underbrace{\sin\frac{1}{x}}_{\text{korlát}} = 0 = f(0) \Rightarrow \text{folytonos}$$

$$f'(x)=\begin{cases} 2x\cdot\sin\frac{1}{x} + x^2\cdot\left(\cos\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right), & \text{ha } x\neq 0\\ 0, \text{ ha } x=0\end{cases}$$

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{h^2\cdot\sin\frac{1}{h}-0}{h}=\lim_{h\to 0}\underset{\downarrow\atop 0}{h}\cdot\underbrace{\sin\frac{1}{h}}_{\text{korl.}}=0$$

$$\lim_{x\to 0} f'(x)=\lim_{x\to 0}\left(2x\cdot\sin\frac{1}{x}-\cos\frac{1}{x}\right)=\nexists \text{ másodfajú szakadás}$$

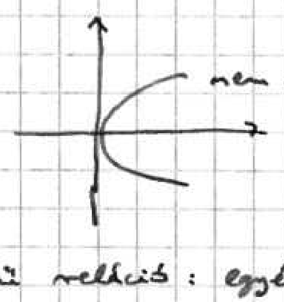
T Egy mindenütt deriválható fv-nél csak másodfajú szakadása lehet.

$$f''(x)=\begin{cases} 2\cdot\sin\frac{1}{x}+2x\cdot\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)-\left(-\sin\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right), & \text{ha } x\neq 0\\ \nexists, \text{ mert } f' \text{ nem folytonos } x=0\text{-ban}\end{cases}$$

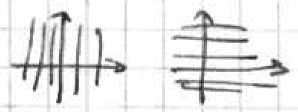
Inverzfüggvény

$A, B \in \mathbb{R}$

$$A \xrightarrow{f} B$$



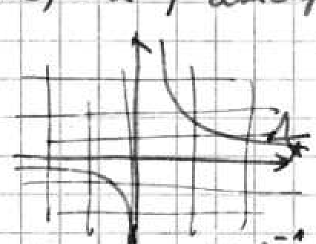
nem fv: nem egyértékű reláció



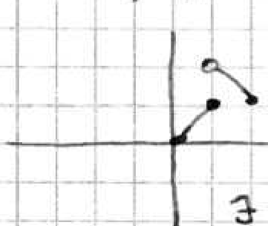
kölcsönösen egyértelmű egyértékű reláció: egyértékű, egyértelmű

D $f^{-1}(b)=a$, amelyre $b=f(a)$ (ez $\forall$ fv-re jó)

pl.

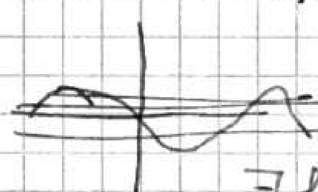


$\exists f^{-1}$

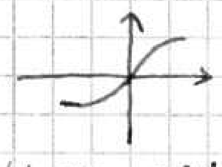


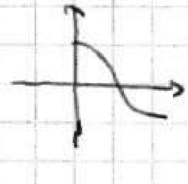
$\exists f^{-1}$

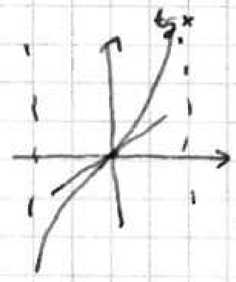
invertálható, és nem szig. mon.

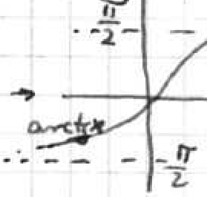


$\exists f^{-1}$ (tükrözés
az origó körül)









Anal. elmélet

2007. 11. 8
Könyv

(Pl) $f(x) = \ln \frac{x^2+1}{x^4+3}$ $\qquad f'(x) = ?$

$$f'(x) = \frac{1}{\frac{x^2+1}{x^4+3}} \cdot \left(\frac{x^2+2}{x^4+3}\right)' = \frac{x^4+3}{x^2+1} \cdot \frac{2x\cdot(x^4+3) - (x^2+2)\cdot 4x^3}{(x^4+3)^2}$$

(értelmezési ~~[illegible]~~ tartomány ugyanaz, és a helyettesítési érték...

Jobb: $f(x) = \ln(x^2+1) - \ln(x^4+3)$

$$f'(x) = \frac{1}{x^2+1}\cdot 2x - \frac{1}{x^4+3}\cdot 4x^3$$

(De: $\ln \frac{x+1}{x-1} \neq \ln(x+1) - \ln(x-1)$, mert az ÉT nem azonos)

$g(x) = \ln x^7$ $\qquad x > 0$

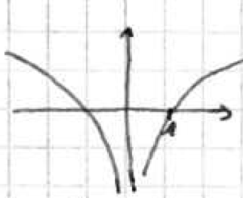
$x > 0$: $g'(x) = \frac{1}{x^7}\cdot 7\cdot x^6 = \frac{7}{x}$

vagy $g(x) = 7\cdot \ln x \Rightarrow g'(x) = 7\cdot\frac{1}{x}$

$\left(\ln x^8 \neq 8\cdot \ln x;\ \ \ln x^8 = 8\cdot \ln|x|\right)$

(Pl) $f(x) = \ln|x|$ $\qquad f'(x) = ?$

$$f(x) = \begin{cases} \ln x, & \text{ha } x > 0 \\ \ln(-x), & \text{ha } x < 0 \end{cases}$$



$$f'(x) = \begin{cases} \frac{1}{x}, & \text{ha } x > 0 \\ \frac{1}{-x}(-1) = \frac{1}{x}, & \text{ha } x < 0 \end{cases}$$

$\left(\Rightarrow \int \frac{1}{x}\,dx = \ln|x| + C \quad C\in\mathbb{R}\right.$
ha $x > 0$ vagy $x < 0$)
kizáró

M: ~~$0^0 := 1$~~ HELYTELEN

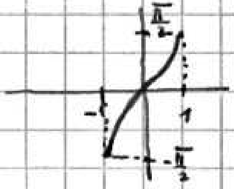
(Pl) $f(x) = (3+\sin^2 5x)^{\frac{1}{\cos^2 x}}$ $\qquad f'(x) = ?$ $\qquad x \neq \frac{\pi}{2} + k\cdot\pi$

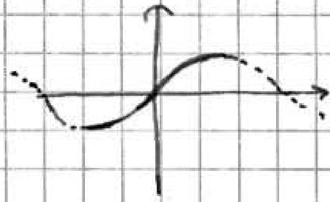
$$f(x) = e^{\ln \dots} = e^{\frac{1}{\cos^2 x}\cdot \ln(3+\sin^2 5x)}$$

$$f'(x) = e^{\frac{\ln(3+\sin^2 5x)}{\cos^2 x}} \cdot \underbrace{\left(\frac{\ln(3+\sin^2 5x)}{\cos^2 x}\right)'}_{\frac{\left(\frac{1}{3+\sin^2 5x}\cdot 2\sin 5x\cdot\cos 5x\cdot 5\right)\cdot\cos^2 x - \ln(3+\sin^2 5x)\cdot 2\cos x\cdot(-\sin x)}{\cos^4 x}}$$

$\boxed{\text{Arcsin}}$

$f(x) = \arcsin x$





$\text{Arcsin } x = \begin{cases} \arcsin x + 2 \\ \pi - \arcsin x \end{cases}$

2k

Anal. elm. 2007.11.14
Kónya

(Pl)
$$f(x)=\begin{cases}\dfrac{x}{7+2^{1/x}}, & \text{ha } x\neq 0\\ 0, & \text{ha } x=0\end{cases}$$

$f'(x)=?$

$$f(0+0)=\lim_{x\to+0}\frac{x^{\to 0}}{7+\underbrace{2^{\frac{1}{x}\to\infty}}_{\to\infty}}\to 0=f(0-0)=\lim_{x\to-0}\frac{x^{\to 0}}{7+\underbrace{2^{\frac{1}{x}\to-\infty}}_{\to 0}}=0=f(0)$$

$$f'(x)=\begin{cases}\dfrac{1\cdot(7+2^{\frac{1}{x}})-x\cdot 2^{\frac{1}{x}}\cdot\ln 2\cdot\left(-\frac{1}{x^2}\right)}{(7+2^{\frac{1}{x}})^2}, & \text{ha } x\neq 0\\ \nexists, & \text{ha } x=0\end{cases}$$

$$\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$$

$$\left(2^{\frac{1}{x}}\right)'=2^{\frac{1}{x}}\cdot\ln 2\cdot\left(-\frac{1}{x^2}\right) \quad (2^x)'=2^x$$

$$\left(\left(\frac{1}{2}\right)^{\frac{1}{x}}\right)'=\left(e^{\ln 2^{\frac{1}{x}}}\right)'=\left(e^{\frac{1}{x}\cdot\ln 2}\right)'=2^{\frac{1}{x}}\cdot\ln 2$$

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim\frac{\frac{h}{7+2^{\frac{1}{h}}}-0}{h}=\lim_{h\to 0}\frac{1}{7+2^{\frac{1}{h}}}$$

$$f'_+(0)=\lim_{h\to+0}\frac{1}{7+\underbrace{2^{1/h}\to\infty}_{\to\infty}}=0$$

$$f'_-(0)=\lim_{h\to-0}\frac{1}{7+\underbrace{2^{1/h}\to-\infty}_{\to 0}}=\frac{1}{7}\neq f'_+(0)$$

$\Rightarrow f'(0)\ \nexists$

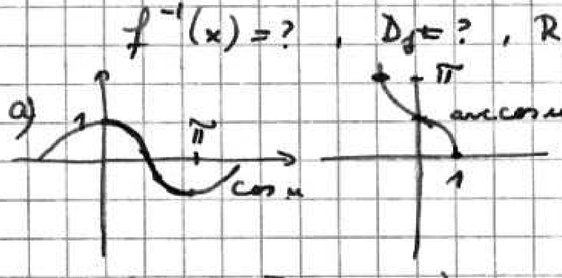
(Pl) $f(x)=\arccos\dfrac{6}{5+x^2}$

a) $D_f=?$ $\quad K_f=?$ $\quad f'(x)=?$

b) Indoklással adja meg azt a legbővebb intervallumot, amelyen $\exists f^{-1}$ és $-2\in I$.

$f^{-1}(x)=?$, $D_{f^{-1}}=?$, $R_{f^{-1}}=?$

a)



$$\underbrace{-1\le\frac{6}{5+x^2}\le 1}$$

$6\le 5+x^2$

$1\le x^2$

$1\le|x|$

$D_f=(-\infty,-1]\cup[1,\infty)$

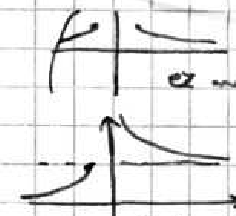
$$\frac{6}{5+x^2}\in(0,1]\Rightarrow\arccos\frac{6}{5+x^2}\in\left[0,\frac{\pi}{2}\right]$$

$$R_f=\left[0,\frac{\pi}{2}\right)\qquad f'(x)=\frac{-1}{\sqrt{1-\left(\frac{6}{5+x^2}\right)^2}}\cdot\underbrace{\left(\frac{6}{5+x^2}\right)'}_{\frac{0-6\cdot 2x}{(5+x^2)^2}}=\frac{12x}{\sqrt{\quad}}\cdot\frac{1}{(5+x^2)^2}$$

$x\in(-\infty,-1)\cup(1,\infty)\quad(|x|>0)$

csak nyílttá kell tenni az intervallumot, különben ~~[illegible]~~ automatikusan >

b) $x \in (-\infty,-1) = I \quad f'(x) < 0 \Rightarrow f$ szig. mon. csökk. $\Rightarrow \exists f^{-1}$ I-n



$I = D_f = (-\infty,-1)$ — ezen a leghosszabb

$R_f = \left(0,\frac{\pi}{2}\right)$ — ezt nyilvánvaló tetten

$y = \arccos \frac{6}{5+x^2}$

$\cos y = \frac{6}{5+x^2}$

$5+x^2 = \frac{6}{\cos y} \qquad x^2 = -5+\frac{6}{\cos y}$

$x = -\sqrt{-5+\frac{6}{\cos y}}$ — I miatt

$x \leftrightarrow y = f^{-1}(x)$

$f^{-1}(x) = -\sqrt{5+\frac{6}{\cos x}}$

$D_{f^{-1}} = R_f = \left(0,\frac{\pi}{2}\right)$

$R_{f^{-1}} = D_f = (-\infty,-1)$

(12) $g(x) = e^{2x} \qquad h(x) = 2x^2+\alpha x+\beta \qquad \alpha = ?, \ \beta = ?$ hogy f mindenütt deriválható legyen.

$f(x) = \begin{cases} g(x), & \text{ha } x \geq 0 \\ h(x), & \text{ha } x < 0 \end{cases}$

$g'(x) = e^{2x} \cdot 2 \qquad h'(x) = 4x+\alpha$

$f'(x) = \begin{cases} g'(x) & x > 0 \\ h'(x) & x < 0 \end{cases}$ 0-ban folytonos? (csatl. pont)

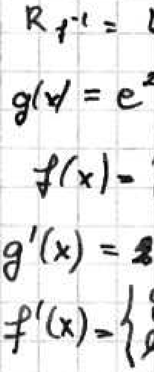
$x = 0$-ban folytonosság: $f(0+0) = g(0+0) = g(0) = 1 = f(0)$

$f(0-0) = h(0-0) = h(0) = \beta = f(0-0)$ $\Rightarrow \beta = 1$

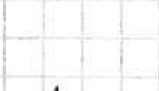
$f'_+(0) = f'_-(0)$

$= g'(0) = 2 \quad = h'(0) = \alpha \Rightarrow \alpha = 2$

(13) $sh(x) = \frac{e^x - e^{-x}}{2} \qquad ch\, x = \frac{e^x + e^{-2x}}{2}$



$(sh\, x)' = ch\, x$

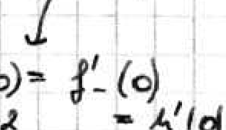


$(ch\, x)' = sh\, x$

$ch^2 x - sh^2 x = 1$

$ch^2 x + sh^2 x = ch\, 2x$

inverz: areash



$ch^2 u - sh^2 u = 1$

$ch\, u = \sqrt{1+sh^2 u}$

$(arsh\, x)' = \left.\frac{1}{ch\, u}\right|_{u = arsh\, x} = \frac{1}{ch\, arsh\, x} = \frac{1}{\sqrt{1+x^2}}$

$(arch\, x)' = \left.\frac{1}{sh\, u}\right|_{u = arch\, x} = \left.\frac{1}{+\sqrt{ch^2 u - 1}}\right|_{u = arch\, x} = \frac{1}{\sqrt{x^2-1}}$

$\frac{1}{\sqrt{1-x^2}} \leftarrow (\arcsin x)'$, mert $\nearrow$

$\frac{-1}{\sqrt{1-x^2}} \leftarrow (\arccos x)'$, mert $\searrow$

$\frac{1}{\sqrt{1+x^2}} = (arsh\, x)'$

$\frac{1}{\sqrt{x^2-1}} = (arsh\, x)'$

2007.11.13.

Anal elm

Pl. $\lim_{x\to+0} \frac{e^{-\frac{1}{x}}}{x} \overset{L'H}{=} \lim_{+0} \frac{e^{-\frac{1}{x}}\cdot\frac{1}{x^2}}{1}$ $\qquad \lim_{+0} \frac{\frac{1}{x}}{e^{\frac{1}{x}}} \overset{L'H}{=} \lim_{+0} \frac{\left(\frac{1}{x}\right)'}{e^{\frac{1}{x}}\cdot\left(\frac{1}{x}\right)'} = 0$

Pl. $\lim_{x\to\pm\infty} \frac{e^{-2x}+2e^{3x}+3}{2e^{-2x}+e^{3x}+5e^{x}}$

$$\lim_{x\to+\infty} \frac{\cancel{e^{3x}}}{\cancel{e^{3x}}}\cdot\frac{e^{-5x}+2+3e^{-3x}}{2\cdot e^{-5x}+1+5\cdot e^{-2x}} = 1\cdot\frac{0+2+0}{0+1+0} = 2$$

$$\lim_{x\to-\infty} \frac{\cancel{e^{-2x}}}{\cancel{e^{-2x}}}\cdot\frac{1+e^{5x}+3e^{2x}}{2+e^{5x}+5e^{3x}} = 1\cdot\frac{1+0+0}{2+0+0} =$$

Pl. $\lim_{x\to\infty} \frac{sh(2x-1)}{ch(2x+3)} = \lim_{\infty}\frac{e^{2x-1}-e^{-2x+1}}{e^{2x+3}+e^{-2x-3}} = \lim_{\infty} \frac{e^{2x}}{e^{2x}}\cdot\frac{e^{-1}-e^{-4x+1}}{e^{3}+e^{-4x-3}} = 1\cdot\frac{e^{-1}-0}{e^{3}+0} = e^{-4}$

Pl. $f(x)=\begin{cases}-2x+\operatorname{arctg}\frac{1}{x-1}, & \text{ha } x\neq 1\\ -2, & \text{ha } x=1\end{cases}$

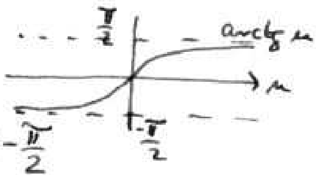
a.) $\lim_{x\to 1+0} f(x)=?$ $\qquad \lim_{x\to 1-0} f(x)=?$ $\qquad \lim_{x\to 1} f(x)=?$

b) $f'(x)=?$, ha $x\neq 1$

$\lim_{x\to 1} f'(x)=?$ $\qquad \exists$-e $f'(1)$?

c) Adja meg azokat a legbővebb intervallumokat, amelyeken az f fv
- szig. mon.
- konvex vagy konkáv

a) $D_f=\mathbb{R}$



$\lim_{x\to 1+0}\left(-2x+\operatorname{arctg}\frac{1}{x-1}\right) = -2+\frac{\pi}{2}$ $\quad \left(\frac{1}{x-1}\right)$: $\frac{1}{+0}\to+\infty$

$\lim_{x\to 1-0}\left(-2x+\operatorname{arctg}\frac{1}{x-1}\right) = -2-\frac{\pi}{2}$ $\quad \frac{1}{-0}\to-\infty$

$\lim_{x\to 1} f(x) \nexists \Rightarrow f$ nem folytonos $x=1$-ben

b) $\underset{x\neq1}{f'(x)} = -2+\frac{1}{1+\left(\frac{1}{x-1}\right)^2}\cdot\left(-\frac{1}{(x-1)^2}\right) = -2-\frac{1}{(x-1)^2+1}$

$\lim_{x\to 1} f'(x) = -2+1=-1$ $\qquad f'(1)\nexists$, mert f nem folyt $x=1$-ben (differenciálhatóság szükséges feltétele)

c.) $f'(x)<0 \Rightarrow$ a fv ~~mindenütt szig~~ szig. mon. csökk $(-\infty,1)$ és $(1,\infty)$ intervallumon

$$f''(x) = -\left((-1)\cdot\frac{1}{((x-1)^2+1)^2}\cdot 2(x-1)\cdot 1\right) = 0 \qquad x\neq 1$$

	$(-\infty,1)$	1	$(1,\infty)$
f''	$-$	$\nexists$	$+$
f	konkáv	szak. h.	konvex

Pl. $f(x) = (1+4x^2)^{\frac{1}{sh\,3x^2}}$ $\qquad x\neq 0$

a) $f'(x)=?$ $\qquad$ b) $\lim_{x\to 0} f(x)=?$

$f(x) = e^{\ln(1+4x^2)^{\frac{1}{sh\,3x^2}}} = e^{\frac{1}{sh\,3x^2}\cdot\ln(1+4x^2)}$

a) $f'(x) = \underbrace{e^{\frac{1}{sh\,3x^2}\cdot\ln(1+4x^2)}}_{(1+4x^2)^{\frac{1}{sh\,3x^2}}}\cdot\left(\frac{\ln(1+4x^2)}{sh\,3x^2}\right)'$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \quad \Bigg\} \quad \frac{\frac{1}{1+4x^2}\cdot 8x \cdot (\mathrm{sh}\, 3x^2) - \ln(1+4x^2)\cdot \mathrm{ch}\, 3x^2 \cdot 6x}{\mathrm{sh}^2 3x^2}$$

b.) $$\lim_{x\to 0} e^{\frac{\ln(1+4x^2)}{\mathrm{sh}\,3x^2}} = e^{4/3}$$

$$\lim_{x\to 0} \underbrace{\frac{\ln(1+4x^2)}{\mathrm{sh}\,3x^2}}_{\frac{0}{0}} = \lim_{x\to 0} \frac{\overset{\to 1}{\frac{1}{1+4x^2}}\cdot 8x}{\underset{\to 1}{\mathrm{ch}\,3x^2}\cdot 6x} = \frac{4}{3}$$

(Pl) $f(x) = (x^2-4x-2)\, e^{2x}$

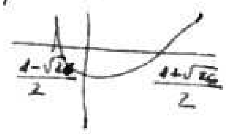
Adja meg azokat a legbővebb nyílt intervallumokat, melyeken a fv
a) szig. mon.
b) konvex, konkáv

$$f'(x) = (2x-4)\, e^{2x} + (x^2-4x-2)\, e^{2x}\cdot 2 = 0$$
$$e^{2x}(2x-4+2x^2-8x-4) = 0$$
$$x^2-3x-4=0$$
$$(x-4)(x+1)=0$$

$f'(x)$: -1, 4

$f(x)$: ↗ lok. max. ↘ lok. min. ↗

$$f''(x) = \left(2e^{2x}\,\cancel{(x-4)}(x+1)\right) = \left(2e^{2x}(x^2-3x-4)\right)' = 2e^{2x}(2x-3) + 2e^{2x}\cdot 2\cdot(x^2-3x-4) = $$
$$\ldots = 2e^{2x}(2x^2-4x-11) = 0$$
$$x = \frac{2\pm\sqrt{26}}{2}$$



		$\frac{1-\sqrt{26}}{2}$		$\frac{1+\sqrt{26}}{2}$	
f''	+	0	−	0	+
$f(x)$	konvex	infl. p	konkáv	infl. p	konvex

(Pl) $$f(x) = \begin{cases} \frac{\sin 2x^2}{5x^x}, & \text{ha } x>0 \\ x\cdot e^{-\frac{1}{(x+3)^2}}, & \text{ha } x<0 \end{cases}$$

$f'(0)\nexists$, mert f nem folytonos ($f(0-0)=0 \neq f(0+0)=\frac{2}{5}$)

$f'(-3)\nexists$, mert nem értelmezett f

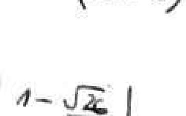
ÉT: $x\neq -3$

szakadási hely: $x=-3$, $x=0$? — véges ugrás; megszüntethető

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$$

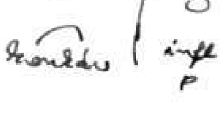
$$f'(x) = \begin{cases} \ldots & x>0 \\ \ldots & x<0 \text{ és } x\neq -3 \end{cases}$$

(Pl) $f(x) = \frac{(x+5)^2}{(x-1)^2}$ Monotonitási szakaszok? $x\neq 1$



$$f'(x) = \frac{2(x+5)\cdot 1\cdot(x-1)^2 - (x+5)^2\cdot 2(x-1)}{(x-1)^4} = \frac{2(x+5)(x-1)[(x-1)-(x+5)]}{\underbrace{(x-1)}_{>0,\ \text{ha } x\neq 1}} = \frac{-12(x+5)(x-1)}{(x-1)^4}$$

		−5		1	
f'	−	0	+	$\nexists$	−
f	↘	lok. min.	↗	szak. h.	↘



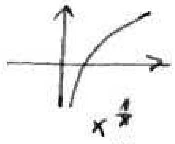
(Pl) $f(x) = \sqrt[x]{x}$ Ábrázolja vázlatosan (f'' vizsgálata nem kell)
Keresse meg az $a_n = \sqrt[n]{n}$ sorozat legnagyobb elemét.

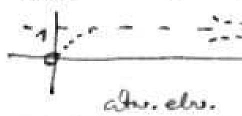
$f(x) = e^{\ln x^{\frac{1}{x}}} = e^{\frac{1}{x}\cdot\ln x}$ $D_f = (0,\infty)$

$$\lim_{x\to 0+0} e^{\frac{1}{x}\ln x} = 0 \qquad \frac{1}{x}\to\infty,\ \ln x\to -\infty$$

$$\lim_{x\to\infty} e^{\frac{1}{x}\ln x} = e^0 = 1$$

$$\lim_{x\to\infty}\frac{\ln x}{x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim \frac{\frac{1}{x}}{1} = 0$$



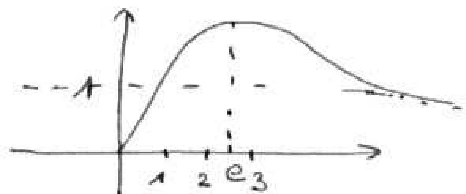


átv. elv. $\boxed{\lim_{n\to\infty}\sqrt[n]{n} = 1}$

$f'(x) = e^{\frac{1}{x}\cdot \ln x} \cdot \left(\frac{\ln x}{x}\right)' = \sqrt[x]{x}\,\frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \underbrace{\sqrt[x]{x}}_{>0} \cdot \frac{1-\ln x}{\underbrace{x^2}_{>0}} = 0 \Rightarrow x = e$

	$(0,e)$	e	(e,∞)
$f'(x)$	$+$	0	$\cancel{+}\ -$
$f(x)$	$\nearrow$	lok. max	$\searrow$

$f(e) = \sqrt[e]{e}$



$R_f = \left(0, \sqrt[e]{e}\right]$

$\max\{\sqrt[n]{n}\} = \max\{\sqrt{2}; \sqrt[3]{3}\} = \sqrt[3]{3}$

Pe a) $f(x) = (2-x)^2\, |\sin(x-2)|$ $f'(x) = ?$

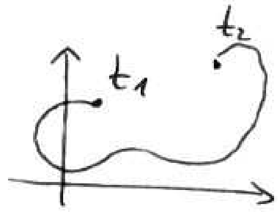
b) $g(x) = (1 + ch^2\, 2x)^{\sin 5x}$

$f(x) = \begin{cases} (2-x)^2 \cdot \sin(x-2) = h, & \text{ha } x \geq 2 \\ -h(x), & \text{ha } x < 2 \end{cases}$ $f'(x) = \begin{cases} h'(x) & x > 2 \\ h(x) & x < 2 \end{cases}$

$f'(2) =$ D - val

Görbék paraméteres megadása

térgörbe
$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$



Már volt egyenes egyenlete $\quad x = 3 - t$, $y = 2 + t$

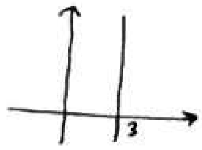
1) Ha a síkgörbe $y = f(x)$ alakban adott, akkor van paraméteres megadható

Ha a síkgörbe $y = f(x)$ alakban adott, —"—

$x = t$
$y = f(t)$ (sőt végtelen sok: $x = t + 4$)

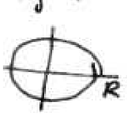
2.) Ha a síkgörbe paraméteresen adott, akkor nem biztos, hogy $\exists$ $y = f(x)$ megadás.

pl: $x = 3$
$y = t$

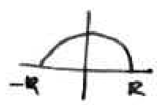


Néhány példa paraméteres megadású görbékre:

1.) $x^2 + y^2 = R^2$

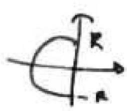


a) $y \geq 0$ $\quad x := t$, $y = \sqrt{R^2 - t^2}$ $\quad t \in [-R, R]$

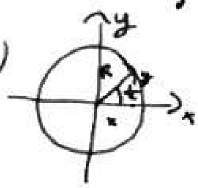


b) $x \leq 0$ $\quad y := t$, $x = -\sqrt{R^2 - t^2}$ $\quad t \in [-R, R]$

$x = -\sqrt{R^2 - y^2}$

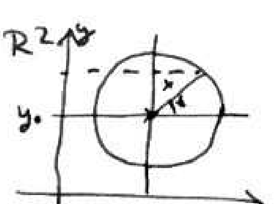


c)



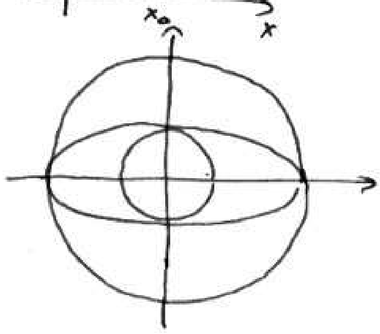
$x = R \cdot \cos t$
$y = R \cdot \sin t$
$t \in [0, 2\pi]$

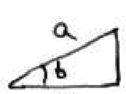
2.) $(x - x_0)^2 + (y - y_0)^2 = R^2$

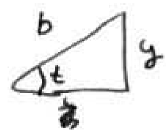


$x = x_0 + R \cdot \cos t$
$y = y_0 + R \sin t$ $\quad t \in [0, 2\pi]$

3.) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$







$x = a \cos t$
$y = b \sin t$ $\quad t \in [0, 2\pi]$

(T) $\quad x = \varphi(t)$
$y = \psi(t)$ $\quad t \in I$

Ha $\varphi(t)$ szigorúan monoton I-n (pl. $\exists\, \dot{\varphi}$ és $\dot{\varphi}(t) \neq 0$ $t \in I$-n vagy csak diszkrét pontban 0), akkor $\exists$ az adott síkgörbének $y = f(x)$ előállítása.

(B) $x=\varphi(t)$ $\exists$ inverze : $t=\varphi^{-1}(x)$: $y=\Psi(t)\Big|_{t=\varphi^{-1}(x)} = \Psi(\varphi^{-1}(t)) = (\Psi\circ\varphi^{-1}$ ...

f =

$\left(\begin{array}{l} x=x(t) \text{ szig. mon} \\ y=y(t) \end{array} \right\} \begin{array}{l} t=t(x) \\ y(t(x))=f(x) \end{array} \Bigg)$

(T) Legyen G görbe paraméteres egyenlete $x=x(t)$, $y=y(t)$, $t\in(t_1,t_2)=I$,
$x(t)$ szigorúan monoton,
$x(t)$, $y(t)$ deriválható $t_0\in I$-ben és $\dot x(t_0)\neq 0$

a) Akkor $f(x)=y(t(x))$ függvény deriválható a megfelelő $x_0=x(t_0)$ pontban és

$$f'(x_0)=\frac{\dot y(t)}{\dot x(t)}\Big|_{t=t_0}=\frac{\dot y(t_0)}{\dot x(t_0)}$$

b.) Ha az előző feltételeken túl $\exists$ $\ddot x(t_0)$ és $\ddot y(t_0)$ is:

$$f''(x_0)=\frac{\ddot y\dot x-\dot y\ddot x}{(\dot x)^3}\Big|_{t_0}$$

(B.) a) $f(x)=y(t(x))$

$$f'(x_0)=\frac{df}{dx}\Big|_{x_0}=\frac{dy(t(x))}{dx}\Big|_{x_0}=\frac{dy}{dt}\Big|_{t_0}\cdot\frac{dt}{dx}\Big|_{x_0}=\frac{\dot y(t_0)}{\dot x(t_0)}$$

$$\frac{dt}{dx}\Big|_{x_0}=\frac{1}{\frac{dx}{dt}}\Big|_{t_0}$$

b.) $$f''(x_0)=\frac{d}{dx}f'(x)\Big|_{x_0}=\frac{d}{dx}\cdot\frac{\dot y(t)}{\dot x(t)}\Big|_{x_0}=\frac{d}{dt}\cdot\frac{\dot y(t)}{\dot x(t)}\Big|_{t_0}\cdot\frac{dt}{dx}\Big|_{x_0}=\frac{\ddot y\cdot\dot x-\dot y\cdot\ddot x}{\dot x^2}\cdot\frac{1}{\left(\frac{dx}{dt}\right)}\Big|_{t_0}=$$

$\left(\frac{dx}{dt}\right) = \dot x$

$$=\frac{\ddot y\dot x-\dot y\cdot\ddot x}{(\dot x)^3}\Big|_{t_0}$$

$$\left(\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\Big|_{t_0}=\left(\leftarrow \text{első derivált}\right)=\frac{\dot y}{\dot x}\Big|_{t_0}=\frac{\ddot y\dot x-\dot y\ddot x}{\dot x^3}\right.$$

tört függvényként kell deriválni

hogy meglegyen a 3 pont a nevezőben

(Fe) $x(t)=e^{2t}+t^3$ $y(t)=\frac{t+1}{t^2+3}$

a) $\dot x(t)=?$ $\dot y(t)=?$

b) Mutassa meg, hogy a fenti par.-es egyenletrendszer a $t_0=0$ paraméterű x_0 pont egy környezetében meghatároz egy $y=f(x)$ fv-t.

c) Lehet-e f-nek lokális szélsőértéke az $x_0=x(t_0)$ pontban?

d) Írja fel a görbe $t_0=0$ pontbeli érintőjének egyenletét Descartes-koordinátákkal

a) $\dot x(t)=e^{2t}\cdot 2+3t^2$ $\dot y(t)=\frac{1\cdot(t^2+3)-(t+1)\cdot 2t}{(t^2+3)^2}$

b) $\dot x(t)>0 \Rightarrow \exists$ $x=x(t)$-nek inverze $\underset{\text{(T) szerint}}{\Rightarrow}$ $y=f(x)=f(t(x))$

c) $x_0=x(0)=1$

$f'(1)=\frac{\dot y(0)}{\dot x(0)}=\frac{\frac{1}{3}}{2}=\frac{1}{6}\neq 0\Rightarrow$ f-nek $x_0=1$-ben nincs lokális szélsőértéke. (szükş. felt. $\emptyset$)

2007. 11. 28.
Kónya

d) $y_é = f(x_0) + f'(x_0)(x - x_0)$

$x_0 = 1 = x(t_0) = x(0)$

$f(x_0) = f(1) = y_0 = y(t_0) = y(0) = 1$

$f'(1) = \frac{1}{6}$

$y_é = 1 + \frac{1}{6} \cdot (x - 1)$

Integrálszámítás

Anal elmélet

2007. 11. 29.
Kónya

Vizsgák: jan. 3., jan. 10., jan. 24., jan. 31.

$$\int \frac{1}{4+x^2}\,dx = \frac{1}{4}\int \frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx = \frac{1}{4}\,\frac{\operatorname{arctg}\frac{x}{2}}{\frac{1}{2}} + C$$

$\left(\int_a^b f(x)\,dx = \right.$  $\left. = F(b) - F(a) = F(x)\Big|_a^b \right)$

$$\int_0^1 \frac{1}{4+x^2}\,dx = \frac{1}{2}\operatorname{arctg}\frac{x}{2} + (2k\ldots)\Big|_0^1 = \frac{1}{2}\left(\operatorname{arctg}\frac{1}{2} - \ldots\right.$$

$$\int \frac{6x}{4+x^2}\,dx = 3\int \underbrace{\frac{2x}{4+x^2}}_{\frac{f'}{f}}\,dx = 3\ln(4+x^2) + C$$

$$\int \frac{6x+1}{4+x^2}\,dx = \frac{1}{2}\operatorname{arctg}\frac{x}{2} + 3\ln(4+x^2) + C$$

$$\int \frac{1}{12+4x+x^2}\,dx = \int \frac{1}{(x+2)^2+8}\,dx = \frac{1}{8}\int \frac{1}{1+\left(\frac{x+2}{\sqrt{8}}\right)^2}\,dx = \frac{1}{8}\cdot\frac{\operatorname{arctg}\frac{x+2}{\sqrt{8}}}{\frac{1}{\sqrt{8}}} + C$$

$\boxed{I = \int \frac{\alpha x+\beta}{ax^2+bx+c}\,dx}$ $\quad D<0$ $\quad$ ($D \geq 0$: részlettörtekre bontás)

$$I = k_1 \underbrace{\int \frac{f'}{f}\,dx}_{\ln|f|} + k_2 \underbrace{\int \frac{1}{f}\,dx}_{\text{teljes négyzetté kiegészítés + kiemelés}}$$

$\dfrac{\operatorname{arctg}(\;)}{(\;)'}$ — lineáris kifejezés.

$$\int \frac{6x}{x^2+2x+3}\,dx = 3\int \underbrace{\frac{2x+2}{x^2+2x+3}}_{\frac{f'}{f}}\,dx - 6\underbrace{\int \frac{1}{x^2+2x+3}}_{\int \frac{1}{(x+1)^2+2}dx = \frac{1}{2}\int \frac{1}{1+\left(\frac{x+1}{\sqrt{2}}\right)^2}dx} = 3\cdot\ln\big(\underbrace{x^2+2x+3}_{\geq 2}\big) - 6\,\frac{\operatorname{arctg}\frac{x+1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} + C$$

$$\boxed{\int \frac{\alpha x+\beta}{\sqrt{ax^2+bx+c}}\,dx} = k_1\int \frac{f'}{\sqrt{f}}\,dx + k_2 \underbrace{\int \frac{1}{\sqrt{f}}\,dx}_{\text{Teljes négyzetté kiegészítés + kiemelés}}$$

$$\int f'\cdot f^{-\frac{1}{2}}\,dx = \frac{f^{+\frac{1}{2}}}{\frac{1}{2}}$$

$$k_3\int \frac{1}{\sqrt{1-(\;)^2}}\,dx = k_3\,\frac{\arcsin(\;)}{(\;)'} \quad \text{lin. kif.}$$

$$k_3\int \frac{1}{\sqrt{1+(\;)^2}}\,dx = k_3\,\frac{\operatorname{arsh}(\;)}{(\;)'}$$

$$k_3\int \frac{1}{\sqrt{(\;)^2+1}}\,dx = k_3\,\frac{\operatorname{arch}(\;)}{(\;)'}$$

Anal. elm

2007. 12. 4.
Könya

racionális törtfüggvény (valódi)

Pl. $\int \frac{2x+1}{x^2-5x+6}\,dx = \int \frac{2x+1}{(x-2)(x-3)}\,dx = \int\left(-5\cdot\frac{1}{x-2} + 7\,\frac{1}{x-3}\right)dx = -5\ln|x-2| + 7\ln|x-3|$

$$\frac{2x+1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3} \quad \Big/ \cdot (x-2)(x-3)$$

1\. mo $\quad 2x+1 = A(x-3) + B(x-2) \qquad \forall\, x$-re

$x=2 \quad 5 = -A \quad \Rightarrow A = -5$
$x=3 \quad 7 = B \quad \Rightarrow B = 7$

2\. mo $\quad 2x+1 = (A+B)\,x + (-3A-2B)$

$$\begin{cases} A+B=2 \\ -3A-2B=1 \end{cases} \quad \ldots$$

Pl. $\int \frac{x+1}{(x-1)^2(x-3)}\,dx = \int\left(-\frac{1}{(x-1)^2} - \frac{1}{x-1} + \frac{1}{x-3}\right)dx = \cancel{\ln(x-1)^2} - \frac{(x-1)^{-1}}{-1} - \ln|x-1| + \ln|x-3| + c$

$$\frac{x+1}{(x-1)^2(x-3)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x-3)} \quad \Big/ \cdot (x-1)^2(x-3)$$

$$x+1 = A(x-3) + B\underbrace{(x-1)(x-3)}_{x^2-4x+3} + C\underbrace{(x-1)^2}_{x^2-2x+1}$$

$x=1 \quad 2 = -2A \qquad A = -1$
$x=3 \quad 4 = 4C \qquad C = 1$
$x=0 \quad 1 = -3A + 3B + C$
$\qquad B = \frac{1}{3}\cdot(1+3A-C) = \frac{1}{3}(1-3-1) = -1 \qquad B = -1$

vagy: együttható öh.

$$x+1 = x^2(B+C) + x(A-4B-2C) + (-3A+3B+C)$$

$$\begin{cases} B = -C \\ A-4B-2C = 1 \\ -3A+3B+C = 1 \end{cases} \quad \ldots$$

Pl. a) $\int \frac{x^3}{x^4-16}\,dx = \frac{1}{4}\int \underbrace{\frac{4x^3}{x^4-16}}_{\frac{f'}{f}} = \frac{1}{4}\cdot\ln|x^4-16| + C$

b) $\int \frac{x^5-15x}{x^4-16}\,dx = ?$

$(x^5-15x) : (x^4-16) = \boxed{x}$
$\underline{x^5-16x}$
$\quad x$

$$\frac{x^5-15}{x^4-16} = x + \frac{x}{x^4-16}$$

Másképp: $\frac{x^5-15x}{x^4-16} = \frac{(x^5-16x)+x}{x^4-16} = x + \frac{x}{x^4-16}$

Anal elmélet 2007.12.5. Kdnya

(PE) $\int \cos 5x \cos 8x \, dx = \frac{1}{2}\int \cos(5x+8x) + \underbrace{\cos(5x-8x)}_{\cos(-3x)=\cos 3x} dx = \frac{1}{2}\left(\frac{\sin 13x}{13} + \frac{\sin 3x}{3}\right) + C$

(PE) $\int \sin^5 x \, dx = \int \sin x \underbrace{(\sin^2 x)^2}_{(1-\cos^2 x)^2 = 1-2\cos^2 x+\cos^4 x} dx = \int \left(\sin x - 2 \underbrace{\sin x \cos^2 x}_{f' \cdot f^2} + \underbrace{\sin x \cdot \cos^4 x}_{f' \cdot f^4}\right) dx =$

$= -\cos x + 2\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} + C$

(PE) $\int \cos^6 x \, dx = \int \underbrace{(\cos^2 x)^3}_{\frac{1+\cos 2x}{2}} dx = \frac{1}{8}\int \left(1 + 3\cos 2x + 3\underbrace{\cos^2 2x}_{\frac{1+\cos 4x}{2}} + \underbrace{\cos^3 2x}_{\cos 2x \cdot (\cos^2 2x)^1 = \cos 2x - \cos 2x \cdot \sin^2 2x}\right) dx$

$1-\sin^2 2x$ $f' \cdot f^2$

$= \frac{1}{8}\left(x + 3 \cdot \frac{\sin 2x}{2} + \frac{3}{2}x + \frac{3}{2} \cdot \frac{\sin 4x}{4} + \frac{\sin 2x}{2} - \frac{1}{2}(-1)\frac{\sin^3 2x}{3}\right) + C$

(PE) $\int \operatorname{arsh} 4x \, dx = \int \underset{\substack{u'=1 \\ u=x}}{1} \cdot \underset{v=\operatorname{arsh} 4x}{\operatorname{arsh} 4x} \, dx = x \cdot \operatorname{arsh} 4x - \int 4x \cdot \frac{1}{\sqrt{1+16x^2}} dx = x \operatorname{arsh} 4x - \frac{1}{8}\frac{(1+16x^2)^{\frac{1}{2}}}{\frac{1}{2}}$

$= \frac{1}{8}\int 32x(1+16x^2)^{-\frac{1}{2}} dx$

(PE) a) $\int (x-3) \cdot e^{2x} dx = \frac{1}{2}(x-3)e^{2x} - \int e^{2x} dx = \frac{1}{2}(x-3)e^{2x} - \frac{1}{2}\frac{e^{2x}}{2} + C$

$u = x-3$, $v' = e^{2x}$, $u' = 1$, $v = \frac{e^{2x}}{2}$

b) $\int (x-3) e^{x^2-6x} dx = \frac{1}{2}\int \underbrace{2(x-3)}_{f'} \underbrace{e^{x^2-6x}}_{e^f} dx = \frac{1}{2}e^{x^2-6x} + C$

RACIONÁLIS TÖRTFÜGGVÉNY

(PE) a) $\int \frac{1}{(x-1)^3} dx$ $f' \cdot f$ b) $\int \frac{1}{x^3-1} dx$ c) $\int \frac{x}{(x-1)^3} dx$ d) $\int \frac{2x^3-1}{(x-1)^3} dx$

e) $\int \frac{2x^2}{x^3-1} dx$ f) $\int \frac{x^4+3}{x^3-1} dx$

a) $\int \underbrace{1 \cdot (x-1)^{-3}}_{f' \cdot f^{-3}} dx = \frac{(x-1)^{-2}}{-2} + C$

b) $\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{(x^2+x+1)} \quad / \cdot (x^3-1)$

$1 = A(x^2+x+1) + (Bx+C)(x-1)$

$1 = (B+A)x^2 + x(A-B+C) + (A-C)$

$$\begin{cases} A+B=0 \\ A-B+C=0 \\ A-C=1 \end{cases} \quad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -2 \end{array}\right) \quad C = -\frac{2}{3},\ B = -\frac{1}{3},\ A = \frac{1}{3}$$

Anal elmélet

2007.12.5.

Könyv

$$\int \frac{\alpha x+\beta}{ax^2+bx+c}\,dx$$

$$= \int k_1 \cdot \frac{f'}{f} + k_2 \cdot \frac{1}{f}$$

$$I_b = \int \left(\frac{1}{3} \cdot \frac{1}{x-1} + \frac{1}{3} \cdot \frac{-x-2}{x^2+x+1} \right) dx = \frac{1}{3} \ln|x-1| + \frac{1}{3}$$

⋮

c) $$\frac{(x-1)+1}{(x-1)^3} = \frac{1}{(x-1)^2} + \frac{1}{(x-1)^3} \qquad I_c = \frac{(x-1)^{-1}}{-1} + \frac{(x-1)^{-2}}{-2} + C$$

d₁) $$\frac{2x^2+1}{(x-1)^3} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^1}$$

⋮

e) $$\int \frac{2x^2}{x^3-1}\,dx = \frac{2}{3} \int \frac{3x^2}{x^3-1}\,dx = \frac{2}{3} \ln|x^3-1| + C$$

$\frac{f'}{f}$

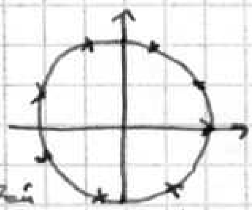
f.) rac. törtfv, általános osztás.

$$(x^4+3) : (x^3-1) = \boxed{x}$$
$$\underline{x^4-x}$$
$$x+3$$

$$\int \boxed{x} + \frac{x+3}{x^3-1}\,dx$$

↑ b.)

$$\int \frac{1}{x^7-1}\,dx \qquad x^7-1=0 \qquad z^7-1=0$$
$$z^7 = 1 \qquad z_{1,\dots,7} = \sqrt[7]{1}$$



3 db valós együtthatójú másodfokú

$$\int \frac{1}{x^8+1}\,dx \qquad \sqrt[8]{-1} \Rightarrow$$ 4 db konjugált komplex gyökökkel ugyanaz, mint az előző

Határozott integrál

- korlátos, zárt intervallumon
- fv korlátos

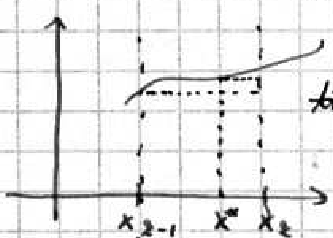
$a < b$

$$m_k = \inf_{x \in I_k} \{ f(x) \}$$

$$M_k = \sup_{x \in I_k} \{ f(x) \}$$

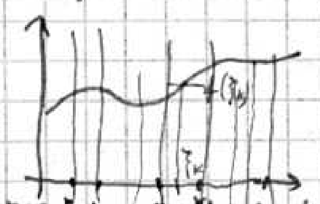
a.ö. $$s_F = \sum_{k=1}^{n} m_k \underbrace{(x_k - x_{k-1})}_{\Delta x_k}$$

f.ö. $$S_F = \sum_{k=1}^{n} M_k \cdot \Delta x_k$$



több részre osztva az s_F nő

- Integrálközelítő összeg



$$\sigma_F = \sum_{k=1}^{n} f(\xi_k)\, \Delta x_k$$

ξ_k: reprezentáns pont (az adott intervallumon)

$f(\xi_k)$

Anal elmélet

2007.12.6.
Kőnya

(P2) $\int_0^1 \frac{\sin x}{x}\,dx$; $\int \frac{1}{x}\,dx$ $\exists$-e ? (Riemann értelemben)

$$f(x)=\begin{cases}\frac{\sin x}{x}, & \text{ha } x\neq 0\\ c, & \text{ha } x=0\end{cases}$$

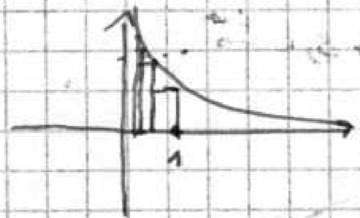
A fv korlátos az int-on, és 1 pont kivételével folytonos
$\Downarrow$
int.ható

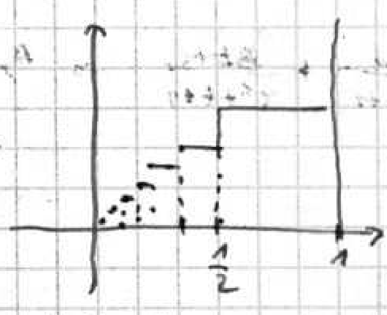
$\exists$-e $\int_0^1 f(x)\,dx$

$$g(x)=\begin{cases}\frac{1}{x}, & \text{ha } x\neq 0\\ \alpha, & \text{ha } x=0\end{cases}$$

$\int_0^1 g(x)\,dx$ $\nexists$, mert g nem korlátos $[0,1]$-ben

(P2) $\int_0^1 f(x)\,dx$ $\exists$-e?





$$f(x)=\begin{cases}\frac{1}{\left[\frac{1}{x}\right]}, & \text{ha } x\neq 0\\ 0, & \text{ha } x=0\end{cases}$$

∞ sok pontban szakadásos.

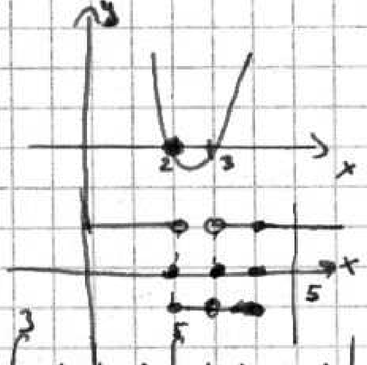
f monoton nő $[0,1]$-en $\overset{(T)}{\Rightarrow}$ $\exists \int_0^1 f(x)\,dx$

Newton-Leibniz tétel

Lagrange-f. k.ét. : $\frac{f(b)-f(a)}{b-a}=f'(\xi) \Rightarrow f(b)-f(a)=f'(\xi)\cdot(b-a)$

$\int_0^5 \operatorname{sgn}(x^2-5x+6)\,dx$

erre nem alkalmazható a Newton-Leibniz-tétel, mert nincs primitív fv



....

$$I=\int_0^2 f+\int_2^3 f+\int_3^5 f=\int_0^2 1\,dx+\int_2^3 -1\,dx+\int_3^5 1\,dx = x\Big|_0^2+(-x)\Big|_2^3+x\Big|_3^5=3$$

Helyettesítés integrálással

(P2) a) $\int \frac{e^{3x}}{e^{3x}+1}\,dx$ $\quad \frac{f'}{f}$

b) $\int \frac{e^{2x}}{e^{3x}+1}\,dx$

Szükség esetén alkalmazza az $e^x=t$ helyettesítést.

a) $\frac{1}{3}\int 3\frac{e^{3x}}{e^{3x}+1}\,dx=\frac{1}{3}\cdot\ln\left(e^{3x}+1\right)+c$

b.) $\int\limits_{x=\varphi(t)} f(x)\,dx = \int f(\varphi(t))\varphi'(t)\,dt \Big|_{t=\varphi^{-1}(x)}$

$$\frac{dx}{dt} = \varphi'(t) \Rightarrow dx = \varphi'(t)\cdot dt$$

$e^x = t$
$x = \ln t \quad \longrightarrow \quad \frac{dx}{dt} = \frac{1}{t}$

$$dx = \frac{1}{t}\,dt$$

I: $\int \frac{t^2}{t^3+1}\cdot\frac{1}{t}\,dt = \int \frac{t}{t^3+1}\,dt =$

$\frac{A}{t+1} + \frac{Bt+C}{t^2+t+1} \rightarrow A, B, C \xrightarrow{\text{int}} \dots$

Anal előadás

2007. 12. 12.
Kónya

(Pl.) $F(x)=\int_0^x \underbrace{e^{t^2+1}}_{\substack{\text{folytonos}\\ \Downarrow \\ \text{deriválható}\\ \forall x\text{-re, int.sz.T. alt.}}} dt \qquad G(x)=\int_0^{x^2} e^{t^2+1}\,dt \qquad H(x)=\int_{2x}^{3x} e^{t^2+1}\,dt$

$$F'(x)=e^{x^2+1}$$

$$G(x)=\int_0^{x^2} e^{t^2+1}\,dt=\left(\int_0^{x^2} e^{x^2+1}\,dx\right)=F(x^2)=\int_0^{x^2} e^{x^2+1}\,dx$$

$$F(x)=\int_0^x e^{x^2+1}\,dx$$

ha mindenütt x helyébe x^2-et írok, az belül nem érinti, csak kívül (belül t^2 [illegible])

$$G'(x)=\frac{d}{dx}F(x^2)=F'(x^2)\cdot 2x=e^{x^4+1}\cdot 2x$$

$$H(x)=\int_0^{3x} e^{t^2+1}\,dt-\int_0^{2x} e^{t^2+1}\,dt=F(3x)-F(2x)$$

$$H'(x)=F'(3x)\cdot 3-F'(2x)\cdot 2=e^{(3x)^2+1}\cdot 3-e^{(2x)^2+1}\cdot 2$$

(Pl.) $f(x)=\int_3^x e^{x^2}(x^2-4)\,dx$

a) Hol monoton f? Hol van lok. szé.-e?

b) Hol konvex, hol konkáv? Hol van inflexiós pontja?

$$f(x)=\int_0^x \underbrace{e^{t^2}(t^2-4)}_{\text{folytonos}}\,dt$$

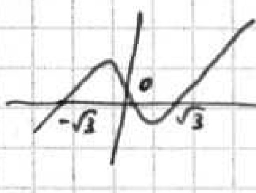
a) $f'(x)=e^{x^2}(x^2-4)=0 \qquad x=\pm 2$

	$(-\infty,-2)$	-2	$(-2,2)$	2	$(2,\infty)$
f'	+	0	−	0	+
f	↗	lok. max	↘	lok. min	↗

lok. max : $f(-2)=\int_3^{-2} e^{t^2}(t^2-4)\,dt$ ezt nem kéri, de nem is tudom kiszámolni

b) $f''(x)=e^{x^2}\cdot 2x(x^2-4)+e^{x^2}\cdot 2x=\underbrace{e^{x^2}}_{>0}\cdot 2x(x^2+3)$

		$-\sqrt{3}$		0		$\sqrt{3}$	
f''	−	0	+	0	−	0	+
f	∩	infl. p.	∪	infl. p.	∩	infl. p.	∪



Anal elm 2007.12.12. Kdyn

(Pl.) $f(x)=\int_0^x x^2\cdot \arcsin x\,dx=\int_0^x t^2\cdot \arcsin t\,dt$

a) Milyen + x-ekre integrálható?

b) $\lim_{x\to 0}\frac{\int_0^x x^2 \arcsin x\,dx}{x}=?$

c) Írja fel a fv $x_0=\frac{1}{2}$ pontbeli érintő egyenesének egyenletét.

a) $0<x<1$, mert ott az integrandus folytonos $\Rightarrow$ az int. sz. II. a.t.-e miatt deriválható
$x<1$-re van a derivált
$x>1$-re nem ért. (a derivált az argumentum felső határon vett értéke)

b) $\frac{0}{0}\ \overset{L'H}{=}\lim_{x\to 0}\frac{x^2\cdot\arcsin x}{1}=0$

c) $y_é=f\left(\frac{1}{2}\right)+f'\left(\frac{1}{2}\right)\cdot\left(x-\frac{1}{2}\right)$

$f'(x)=x^2\arcsin x$

$f'\left(\frac{1}{2}\right)=\frac{1}{4}\cdot\arcsin\frac{1}{2}=\frac{1}{4}\cdot\frac{\pi}{6}$

$f\left(\frac{1}{2}\right)=\int_0^{1/2} t^2\cdot\arcsin t\,dt=\ldots$

(Pl.) $\int_2^4\sqrt{x^2-6x+9}=\int_2^4\sqrt{(x-3)^2}=\int_2^4|x-3|=\int_2^3-(x-3)\,dx+\int_3^4(x-3)\,dx=\ldots$

$\int_{-2}^3|x^3-x^2|\,dx=\int_{-2}^3 x^2|x-1|\,dx=\int_{-2}^1 x^2(1-x)\,dx+\int_1^3 x^2(x-1)\,dx=\left.\frac{x^3}{3}-\frac{x^4}{4}\right|_{-2}^1+\frac{x^4}{4}-\frac{x^3}{3}\Big|$

(Pl.) a) $\frac{d}{dx}\int_a^b\cos(x^3+1)\,dx$ b.) $\frac{d}{da}\int_a^b\cos(x^3+1)\,dx$ c) $\frac{d}{db}\int_a^b\cos(x^3+1)\,dx$

a) $\frac{d}{dx}\int_a^b\cos(x+1)\,dx=\frac{d}{dx}\sin(x+1)\Big|_a^b=\frac{d}{dx}\left(\sin(b+1)-\sin(a+1)\right)=0$

$\frac{d}{dx}\int_a^b\cos(x^3+1)\,dx=\int_a^b\cos(u^3+1)\,du=0$

Anal elmélet

2007. 12. 12.
Kónya

(Pl) a) $\int \frac{1}{\sqrt[3]{x-2}}\,dx = ?$ $f' \cdot f^{\alpha}$ b.) $\int \frac{x}{\sqrt[3]{x-2}}\,dx$ Szükség esetén alkalmazzuk a $\sqrt[3]{x-2} = t$ helyettesítést.

a) $\int 1 \cdot (x-2)^{-\frac{1}{3}}\,dx = \frac{(x-2)^{2/3}}{\frac{2}{3}} + C$

b) $\sqrt[3]{x-2} = t$

$x = t^3 + 2 = \varphi(t)$

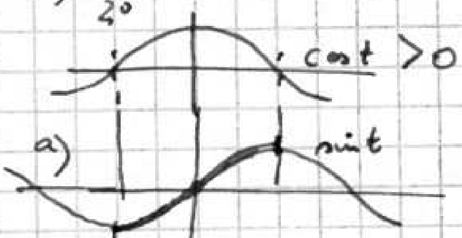
$\frac{dx}{dt} = 3t^2 \Rightarrow dx = 3t^2 \cdot dt$

$I_b: \int \frac{t^3+2}{t} \cdot 3t^2\,dt = \int 3t^4 + 6t\,dt = \frac{3 \cdot t^5}{5} + \frac{6 \cdot t^2}{2} + C$

$I_b = \frac{3}{5}\left(\sqrt[3]{x-2}\right)^5 + 3 \cdot \left(\sqrt[3]{x-2}\right)^2 + C$

(Pl) a) $\int_0^{1/\sqrt{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx = ?$ b.) $\int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^2}}\,dx$ Szükség esetén $x = \sin t$

b) $-\frac{1}{2}\int_0^{1/\sqrt{2}} -2x \cdot (1-x^2)^{-\frac{1}{2}}\,dx = -\frac{1}{2}\,\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\Big|_0^{\frac{1}{\sqrt{2}}} = \ldots$



$t \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

$x = \sin t$ $x = a = 0$ $x = b = \frac{1}{\sqrt{2}}$

$dx = \cos t\,dt$ $0 = \sin\alpha \Rightarrow \alpha = 0$ $\frac{1}{\sqrt{2}} = \sin\beta \Rightarrow \beta = \frac{\pi}{4}$

$I = \int_0^{\frac{\pi}{4}} \underbrace{\frac{\sin^2 t}{\sqrt{1-\sin^2 t}}} \cdot \cos t\,dt = \int_0^{\frac{\pi}{4}} \frac{\sin^2 t}{\cos t} \cdot \cos t\,dt = \int_0^{\frac{\pi}{4}} \frac{1-\cos 2t}{2}\,dt = \ldots$

$\frac{\sin^2 t}{\sqrt{\cos^2 t}} = \frac{\sin^2 t}{|\cos t|}$

(Pl) $\int \frac{e^{2x} + 5e^x}{e^{2x} + 4e^x + 3}\,dx = ?$ $e^x = t$

$x = \ln t$ $I: \int \frac{t^2+5t}{t^2+4t+3} \cdot \frac{1}{t}\,dt = \int \frac{t+5}{(t+1)(t+3)}\,dt = \ldots$

$dx = \frac{1}{t}\,dt$

$\frac{A}{t+1} + \frac{B}{t+3} = \ldots$