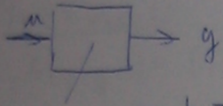


$$D(z) = \frac{b_1 z + b_2}{z^2 + a_1 z + a_2}$$

$$D(z^{-1}) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{b_1 + b_2 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} z^{-1} = \frac{Y(z)}{U(z)}$$

$$(1 + a_1 z^{-1} + a_2 z^{-2}) Y = (b_1 + b_2 z^{-1}) z^{-1} U$$

$$y(t) + a_1 y(t-1) + a_2 y(t-2) = b_1 u(t-1) + b_2 u(t-2)$$

$$x(t); \quad t = \ldots, -2, -1, 0, 1, 2, \ldots$$

$$q^{-1} x(t) = x(t-1), \quad q^{-k} x(t) = x(t-k)$$

$$A(q) = 1 + a_1 q^{-1} + a_2 q^{-2}$$

$$B(q) = b_1 + b_2 q^{-1}$$

$$\boxed{A(q)\, y(t) = B(q)\, u(t-1)}$$

$$\left.\begin{array}{l} y(t) = -a_1 y(t-1) - a_2 y(t-2) + b_1 u(t-1) + b_2 u(t-2) \\ \vartheta = (a_1 \;\; a_2 \;\; b_1 \;\; b_2) \\ \varphi^T(t) = [-y(t-1) \;\; -y(t-2) \;\; u(t-1) \;\; u(t-2)] \end{array}\right\} \boxed{y(t) = \varphi^T(t) \cdot \vartheta}\,; \; \vartheta = ?$$

lineáris paraméter becslési probléma

<u>Autoregresszív</u> folyamat (AR):

$$A(q)\, y(t) = e(t)$$

fehér zaj: frekvencia spektruma konstans; Normál eloszlás $N(0, \sigma)$

$$(1 + a_1 q^{-1} + \ldots + a_{n_a} q^{-n_a})\, y(t) = e(t)$$

$$y(t) + a_1 y(t-1) + \ldots + a_{n_a} y(t - n_a) = e(t)$$

<u>Mozgóátlag</u> folyamat (MA):

$$y(t) = C(q)\, e(t)$$

$$y(t) = (1 + c_1 q^{-1} + \ldots + c_{n_c} q^{-n_c})\, e(t)$$

$$y(t) = e(t) + c_1 e(t-1) + \ldots + c_{n_c} e(t - n_c)$$

Külső jel (exogenous signal) X:

$$B(q)\,u(t) = (b_1 + b_2 q^{-1} + \ldots + b_{n_b} q^{-(n_b-1)})\,u(t) =$$
$$= b_1 u(t) + b_2 u(t-1) + \ldots + b_{n_b} u(t - n_b + 1)$$

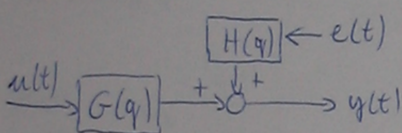
ARX modell:

$$A(q)\,y(t) = B(q)\,u(t - \underset{\substack{\uparrow \\ \text{holt idő}}}{n_k}) + e(t)$$

ARMAX modell:

$$A(q)\,y(t) = B(q)\,u(t-n_k) + C(q)\,e(t)$$
$$\Rightarrow y(t) = \frac{B(q)\,q^{-n_k}}{A(q)}\,u(t) + \frac{C(q)}{A(q)}\,e(t)$$

Általánosabb eset:



$$y(t) = G(q)\,u(t) + H(q)\,e(t)$$

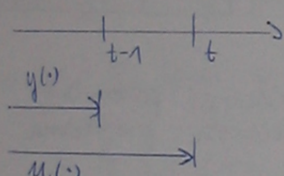
Sorba fejtés:

$$H(q) = 1 + \underbrace{h_1 q^{-1} + h_2 q^{-2} + \ldots} = 1 + q^{-1}\,\tilde{H}(q)$$

$$\tilde{H} = q\,(H-1)$$

$$y(t-k) = G(q)\,u(t-k) + H(q)\,e(t-k) \Rightarrow$$
$$e(t-k) = H^{-1}(q)\,[y(t-k) - G(q)\,u(t-k)]$$
$$y(t) = G(q)\cdot u(t) + e(t) + q^{-1}\,\tilde{H}(q)\cdot$$
$$\cdot \underbrace{H^{-1}(q)\,[y(t) - G(q)\,u(t)]}_{e(t)}$$

Előre jóslás:



$$(y^{t-1}_{-\infty},\ u^{t}_{-\infty}) \to \hat{y}(t\,|\,t-1)$$

$$y(t) = [1 - q^{-1}\tilde{H}(q)\,H^{-1}(q)]\,G(q)\,u(t) + q^{-1}\tilde{H}(q)\,H^{-1}(q)\,y(t) + e(t)$$

$$\boxed{\hat{y}(t\,|\,t-1) = q^{-1}\tilde{H}(q)\,H^{-1}(q)\,y(t) + [1 - q^{-1}\tilde{H}(q)\,H^{-1}(q)]\,G(q)\,u(t)}$$

$$\boxed{\varepsilon(t) = y(t) - \hat{y}(t\,|\,t-1) = H^{-1}(q)\,[y(t) - G(q)\,u(t)]}$$

$$\tilde{H} = q\,(H-1) \Rightarrow q^{-1}\tilde{H}H^{-1} = \cancel{q^{-1}}\cancel{q}\,(H-1)\,H^{-1} = [1 - H^{-1}]$$

$$1 - q^{-1}\tilde{H}H^{-1} = \cancel{1} - (\cancel{1} - H^{-1}) = H^{-1}$$

$$\boxed{\underset{\substack{\uparrow \\ \text{jóslás}}}{\hat{y}}(t\,|\,t-1) = H^{-1}(q)\,G(q)\,u(t) + [1 - H^{-1}(q)]\,y(t)}$$

ARX modell

$$A(q)\,y(t) = B(q)\,u(t) + e(t) \qquad G = \frac{B}{A}\ ;\quad H = \frac{1}{A} \Rightarrow H^{-1} = A$$

$$y(t) = \frac{B(q)}{A(q)}\,u(t) + \frac{1}{A(q)}\,e(t) \qquad H^{-1}G = A\,\frac{B}{A} = B\ ;\quad 1 - H^{-1} = 1 - A$$

$$\hat{y}(t|t-1) = [1 - A(q)]\,y(t) + B(q)\,u(t)$$

$$A = 1 + a_1 q^{-1} + \dots + a_{n_a} q^{-n_a}$$

$$B = b_1 + b_2 q^{-1} + \dots + b_{n_b} q^{-(n_b-1)}$$

$$\hat{y}(t|t-1) = \underbrace{[-y(t-1) \dots -y(t-n_a)\; u(t) \dots u(t-n_b+1)]}_{\varphi^T(t)} \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_{n_a} \\ b_1 \\ \vdots \\ b_{n_b} \end{pmatrix}}_{\vartheta}$$

$$y(t) = \varphi^T(t)\,\vartheta\ , \quad t = 1, \dots, N$$

Veszteség fv.: $V = \frac{1}{2} \sum_{t=1}^{N} \left[y(t) - \varphi^T(t)\,\vartheta \right]^2 \longrightarrow$ minimalizálni

Jósolt értékek: $Y = \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix}$; $\Phi = \begin{bmatrix} \varphi^T(1) \\ \vdots \\ \varphi^T(N) \end{bmatrix}$

$$V = \frac{1}{2} \| Y - \Phi\,\vartheta \|^2 \longrightarrow \min.$$

$$V = \frac{1}{2} \langle Y - \Phi\vartheta, Y - \Phi\vartheta \rangle = \frac{1}{2} \left\{ \langle Y, Y \rangle - 2 \langle Y, \Phi\vartheta \rangle + \langle \Phi\vartheta, \Phi\vartheta \rangle \right\}$$

$$= \frac{1}{2} \left\{ \langle Y, Y \rangle - 2 \langle \Phi^T Y, \vartheta \rangle + \langle \Phi^T \Phi\,\vartheta, \vartheta \rangle \right\}$$

skaláris szorzatok

$$\frac{dV}{d\vartheta} = \operatorname{grad} V = \frac{1}{2} \left\{ -2\,\Phi^T Y + 2\,\Phi^T \Phi\,\vartheta \right\} = 0$$

(1) $\Phi^T \Phi\,\vartheta' = \Phi^T Y$ normál egyenlet

(2) $\vartheta' = [\Phi^T \Phi]^{-1} \Phi^T Y$ LS-optimum

Általánosítása: Súlyozott LS módszer

$W > 0$ (szimmetrikus és pozitív definit)

$$V = \frac{1}{2} \langle \underbrace{W(Y - \Phi\vartheta)}_{WY - W\Phi\vartheta}, Y - \Phi\vartheta \rangle = \frac{1}{2} \left\{ \langle WY, Y \rangle - \langle WY, \Phi\vartheta \rangle - \langle W\Phi\vartheta, Y \rangle + \langle W\Phi\vartheta, \Phi\vartheta \rangle \right\} =$$

$$= \frac{1}{2}\{\langle WY, Y\rangle - 2\langle \Phi^T W Y, \vartheta\rangle + \langle \Phi^T W \Phi \vartheta, \vartheta\rangle\}$$

$$\frac{dV}{d\vartheta} = \text{grad}\, V = \frac{1}{2}\{-2\Phi^T W Y + 2\Phi^T W \Phi \vartheta\} = 0$$

(1) $\Phi^T W \Phi \hat{\vartheta} = \Phi^T W Y$ normál egyenlet

(2) $\hat{\vartheta} = [\Phi^T W \Phi]^{-1} \Phi^T W Y$ WLS-módszer

$$\hat{\vartheta}^{LS} = [\Phi^T \Phi]^{-1} \Phi^T Y$$

$$\Phi^T \Phi = [\varphi(1) \dots \varphi(N)] \begin{bmatrix} \varphi^T(1) \\ \vdots \\ \varphi^T(N) \end{bmatrix} = \sum_{t=1}^{N} \varphi(t)\varphi^T(t)$$

$$\Phi^T Y = [\varphi(1) \dots \varphi(N)] \begin{bmatrix} y(1) \\ \vdots \\ y(N) \end{bmatrix} = \sum_{t=1}^{N} \varphi(t) \cdot y(t)$$

$$\hat{\vartheta}^{LS} = \left[\sum_{t=1}^{N} \varphi(t)\cdot\varphi^T(t)\right]^{-1} \cdot \left[\sum_{t=1}^{N} \varphi(t)\, y(t)\right] = \left[\frac{1}{N}\sum_{t=1}^{N} \varphi(t)\varphi^T(t)\right]^{-1} \cdot \left[\frac{1}{N}\sum_{t=1}^{N} \varphi(t) y(t)\right]$$

<u>Példa</u>: $y(t) = \varphi^T(t)\vartheta_0 + \nu_0(t)$

$$\hat{\vartheta}^{LS} = \left[\frac{1}{N}\sum_{t=1}^{N} \varphi(t)\varphi^T(t)\right]^{-1} \left[\frac{1}{N}\sum_{t=1}^{N} \left(\varphi^T(t)\vartheta_0 + \nu_0(t)\right)\varphi(t)\right]$$

$$\hat{\vartheta}^{LS} = \vartheta_0 + \left[\frac{1}{N}\sum_{t=1}^{N} \varphi(t)\cdot\varphi^T(t)\right]^{-1} \left[\frac{1}{N}\sum_{t=1}^{N} \varphi(t)\cdot\nu_0(t)\right]$$

legyenek függetlenek

Ehhez módszer kell!

Segédváltozók módszere (instrumental variables) IV:

$$\frac{1}{N}\sum_{t=1}^{N} \varphi(t)\varphi^T(t) = R_N \longrightarrow R_{\varphi\varphi} \text{ autokorrelációs fv.}$$

$$\frac{1}{N}\sum_{t=1}^{N} \varphi(t)\nu_0(t) = h_N \longrightarrow h_{\varphi\nu_0} \text{ kereszt korr. fv.}$$

$$\hat{\vartheta}^{LS} = \vartheta_0 + R_N^{-1} h_N$$

$$\hat{\vartheta}^{LS} = \text{solve}\left\{\frac{1}{N}\sum_{t=1}^{N} \varphi(t)\left[y(t) - \varphi^T(t)\vartheta\right] = 0\right\}$$

helyette egy segédváltozó: $\xi(t)$

$$\hat{\vartheta}^{IV} = \text{solve}\left\{ \frac{1}{N}\sum_{t=1}^{N} \xi(t)\,[y(t) - \varphi^T(t)\vartheta] = 0 \right\}$$

$$\hat{\vartheta}^{IV} = \left[\frac{1}{N}\sum_{t=1}^{N} \xi(t)\,\varphi^T(t)\right]^{-1} \left[\frac{1}{N}\sum_{t=1}^{N} \xi(t)\,y(t)\right]$$ $\wedge$ = becsült

Elvárások: $\frac{1}{N}\sum_{t=1}^{N} \xi(t)\,\varphi^T(t)$ legyen invertálható

$$\frac{1}{N}\sum_{t=1}^{N} \xi(t)\,v_0(t) = 0$$

4 lépéses IV algoritmus: IV4 módszer

① LS becslés $\longrightarrow \hat{\vartheta}^{LS} = \hat{\vartheta}^1 \longrightarrow \hat{B}^1(q) / \hat{A}_1(q) = \hat{G}_1(q)$

② $x(t) = \hat{G}_1(q)\,u(t)$

$$\xi^1(t) = [-x(t-1) \dots -x(t-n_a)\; u(t) \dots u(t-n_b+1)]^T$$

$$\hat{\vartheta}^{IV} = \hat{\vartheta}^2 \longrightarrow \hat{B}_2(q)/\hat{A}_2(q) = \hat{G}_2(q)$$

③ $\hat{w}^2(t) = \hat{A}_2(q)\,y(t) - \hat{B}_2(q)\,u(t)$

$\underbrace{L(q)}_{\text{fehérítő szűrő}}\,\hat{w}^2(t) = e(t)$ ← fehér zaj. ← AR modell, qr L nagy $\rightarrow \hat{L}(q)$

④ $x^2(t) = \hat{G}_2(q)\,u(t)$

$$\xi^2(t) = \hat{L}(q)\,[-x^2(t-1) - \dots - x^2(t-n_a)\; u(t) \dots u(t-n_b+1)]^T$$

$$\varphi_F(t) = \hat{L}(q)\,\varphi(t); \quad y_F(t) = \hat{L}(q)\,y(t)$$

Végül (① → ④):

$$\hat{\vartheta}^{IV} = \left[\frac{1}{N}\sum_{t=1}^{N} \xi^2(t)\,\varphi_F^T(t)\right]^{-1} \left[\frac{1}{N}\sum_{t=1}^{N} \xi^2(t)\,y_F(t)\right]$$

$$\hat{\vartheta}^{IV} = \hat{\vartheta}^3 \rightarrow \hat{B}_3(q) / \hat{A}_3(q) = \hat{G}_3(q)$$

(AR) $A(q)\,y(t) = e(t)$ — nn = na

(ARX) $A(q)\,y(t) = B(q)\,u(t-n_k) + e(t)$ — nn = [na nb nk]

(ARMAX) $A(q)\,y(t) = B(q)\,u(t-n_k) + C(q)\,e(t)$ — nn = [na nb nc nk]

$\vdots$ $\rightarrow$

Kimeneti hiba modell (OE): $y(t) = \frac{B(q)}{F(q)} u(t-nk) + e(t)$

Box Jenkins m. (BJ): $y(t) = \frac{B(q)}{F(q)} u(t-nk) + \frac{C(q)}{D(q)} e(t)$

Paraméter becslési m. (PEM): $A(q) y(t) = \frac{B(q)}{F(q)} u(t-nk) + \frac{C(q)}{D(q)} e(t)$

$Z = [y \quad u]_{N \times 2}$

$nn = [na \ nb \ nc \ nd \ nf \ nk]$

(AR): ar(y, nn) = $\underline{\text{thar}}$ ⇐ th struktúra

(ARX): tharx=arx(Z, nn)

thiv4 = iv4(Z, nn)

(ARMAX): tharmax = armax(Z, nn)

⋮

(PEM): thpem = pem(Z, nn) ⇒ [A, B, C, D, F] = th2poly(thpem)

Armax ⟶ kvázi - Newton módszer

Newton módszer:

$$f(x) = \frac{1}{2} \langle A(x-x_0), x-x_0 \rangle \quad ; \quad A > 0 \text{ (poz. definit)}$$

$$f'(x) = \text{grad} f = \cancel{\frac{1}{2}} \cancel{2} A(x-x_0)$$

$$f''(x) = A \quad \text{(Hess mátrix)}$$

$$x_0 = x - A^{-1}[A(x-x_0)] = x - [f''(x)]^{-1} f'(x)$$

$$\boxed{x_{new} = x_{old} - [f''(x_{old})]^{-1} f'(x_{old})}$$

nehezen számítható, ezért közelíteni kell:

$$f''(x) \approx H(x) \Rightarrow$$

kvázi:

$$\boxed{x_{new} = x_{old} - [H(old)]^{-1} f'(x_{old})}$$

$V = $ → V' számítható

→ $V'' = \underbrace{1.\text{ tag}}_{\substack{\approx 0 \\ \text{optimum} \\ \text{közelében}}} + \underbrace{2.\text{ tag}}_{\substack{\text{csak } f'(x) \\ \text{kell hozzá}}}$

armax ⟷ kvázi Newton - mód.