—1—

1, a, $\alpha := \overline{\lim} \sqrt[n]{|a_n|}$; $R = \begin{cases} 0, & \text{ha } \alpha = \infty \\ \frac{1}{\alpha}, & \text{ha } \alpha > 0 \\ \infty, & \text{ha } \alpha = 0 \end{cases}$ (a hatványsor: $\sum_{n=0}^{\infty} a_n x^n$)

[4]

vagy $\alpha := \lim \left| \frac{a_{n+1}}{a_n} \right|$

b, [9] $\sum_{n=0}^{\infty} \underbrace{\left(\frac{8n^2+2n+3}{(2n+3)^2} \right)^n}_{a_n} x^{2n} = \sum_{n=0}^{\infty} a_n y^n$

Legyen $y = x^2$ ②

$\alpha_y = \overline{\lim} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{8n^2+2n+3}{(2n+3)^2} = \frac{8}{4} = 2$ ④ $\Rightarrow R_y = \frac{1}{2}$ ①

Így $R_x = \sqrt{R_y} = \underline{\underline{\frac{1}{\sqrt{2}}}}$ ②

— o —

2, $f(x) = e^{5x} = \sum_{n=0}^{\infty} \frac{(5x)^n}{n!} = \underline{\underline{\sum_{n=0}^{\infty} \frac{5^n}{n!} x^n}}$ ④, hiszen $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

$e^{5x} = e^{5(x-2)+10} = \underline{\underline{e^{10} \sum_{n=0}^{\infty} \frac{5^n}{n!} (x-2)^n}}$ ⑥

— o —

3, a, [8] $f(x) = \frac{1}{2+3x} = \frac{1}{2} \cdot \frac{1}{1-\left(-\frac{3}{2}x\right)}$ ③ $= \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{3}{2}x\right)^n$ ③ $= \sum_{n=0}^{\infty} \frac{(-3)^n}{2^{n+1}} x^n$

↑ Ha $\left|-\frac{3}{2}x\right| < 1$, azaz ha $|x| < \frac{2}{3}$

Tehát $R = \frac{2}{3}$. ②

b, [9] $g(x) = \ln(2+3x)$ Az előző sorfejtést a konvergencia-tartományon belül tagonként integrálhatjuk: ②

$\int_{t=0}^{x} f(t)\,dt = \int_{t=0}^{x} \frac{1}{2+3t}\,dt = \frac{\overbrace{\ln(2+3x)}^{g(x)} - \ln 2}{3} = \sum_{n=0}^{\infty} \frac{(-3)^n}{2^{n+1}} \cdot \frac{x^{n+1}}{n+1}$ ③

Tehát $g(x) = \cancel{x}\cdot \ln 2 + \sum_{n=1}^{\infty} \frac{(-3)^{n \cancel{+1}}}{2^n \cdot n} x^n$ ②, $R = \frac{2}{3}$ ② (nem változik)

4, $f(x) = \sqrt[3]{(8-2x^3)^2}$; $x_0 = 0$

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$$f(x) = (8-2x^3)^{2/3} = 8^{2/3}\left(1 - \frac{x^3}{4}\right)^{2/3} = 4\sum_{k=0}^{\infty}\binom{2/3}{k}\left(\frac{-x^3}{4}\right)^k =$$ (3)

ha $\left|\frac{x^3}{4}\right| < 1$, azaz

$$= \sum_{k=0}^{\infty}\underbrace{\frac{(-1)^k}{4^{k-1}}\binom{2/3}{k}}_{a_{3k}} x^{3k}$$ (3)

$R = \sqrt[3]{4}$ (2)

$$f^{(9)}(0) = 9! \cdot a_9 = 9!\,\frac{(-1)^3}{4^2}\cdot\frac{2/3\cdot(-1/3)\cdot(-4/3)}{3!}$$ (2) (3)

$3k = 9,\ k = 3$

$f^{10}(0) = 0$, (2) hiszen csak a $3k$ alakú hatványkitevők szerepelnek.

5,

13

$$f(x,y) = \begin{cases} \dfrac{3x^2+6y^2+xy}{x^2+2y^2}, & \text{ha } (x,y) \neq (0,0) \\ 3, & \text{ha } (x,y) = (0,0) \end{cases}$$

Legyen $x = r\cos\varphi$, $y = r\sin\varphi$ (3)

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0}\frac{3r^2\cos^2\varphi + 6r^2\sin^2\varphi + r^2\sin\varphi\cos\varphi}{r^2\cos^2\varphi + 2r^2\sin^2\varphi} =$$

$$= \lim_{r\to 0}\frac{3 + 3\sin^2\varphi + \sin\varphi\cos\varphi}{1+\sin^2\varphi} = \nexists$$ (3) (függ φ-től)

(2)

Tehát f-nek nem létezik határértéke az origóban $\Rightarrow$ ott nem is folytonos. (2) A többi pontban f folytonos, mert folyt. fgv.-ek hányadosa, és a nevező $\neq 0$. (3)

6, [18] $f(x,y)=\begin{cases}\dfrac{2x^2y}{x^2+y^2}, & \text{ha } (x,y)\neq(0,0)\\ 0, & \text{ha } (x,y)=(0,0)\end{cases}$

[5] a, Legyen $x = r\cos\varphi$, $y = r\sin\varphi$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0}\frac{2r^3\cos^2\varphi\sin\varphi}{r^2} = 0 = f(0,0)$$ ③, tehát f folyt. az origóban. ②

b, Ha $(x,y)\neq(0,0)$:

[9] [11] $$f'_x(x,y)=\frac{4xy(x^2+y^2)-2x^2y\cdot 2x}{(x^2+y^2)^2}$$ ③

$$f'_y(x,y)=\frac{2x^2(x^2+y^2)-2x^2y\cdot 2y}{(x^2+y^2)^2}$$ ③

Ha $(x,y)=(0,0)$:

$$f'_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$ ~~②~~ ③ Hasonlóan $f'_y(0,0)=0$ ~~①~~ ②

[4] [2] c, Az origón kívül $\exists$ grad f, mert a parc. deriváltak léteznek és folytonosak $\mathbb{R}^2\setminus(0,0)$-n. ② Az origóban $\nexists$ grad f, mert itt f ~~nem~~ folyt. ~~②~~

— o —

7, [14] $f(x,y)=g\left(\frac{2x}{1+y^2}\right)$

$$f'_x(x,y)=g'\left(\frac{2x}{1+y^2}\right)\cdot\frac{2}{1+y^2}$$ ②; $$f'_y(x,y)=g'\left(\frac{2x}{1+y^2}\right)\cdot\frac{-2x\cdot 2y}{(1+y^2)^2}$$ ③

$$f''_{xx}(x,y)=g''\left(\frac{2x}{1+y^2}\right)\cdot\left(\frac{2}{1+y^2}\right)^2$$ ②;

$$f''_{yy}(x,y)=g''\left(\frac{2x}{1+y^2}\right)\cdot\frac{(4xy)^2}{(1+y^2)^4}+g'\left(\frac{2x}{1+y^2}\right)\cdot\frac{-4x(1+y^2)^2+4(xy)\cdot 2(1+y^2)\cdot 2y}{(1+y^2)^4}$$ ③

$$f''_{xy}(x,y)=f''_{y,x}(x,y)=g''\left(\frac{2x}{1+y^2}\right)\cdot\frac{2}{1+y^2}\cdot\frac{-4xy}{(1+y^2)^2}+g'\left(\frac{2x}{1+y^2}\right)\cdot\frac{-4y}{(1+y^2)^2}$$ ③

↑ ①

PÓTFELADATOK:

8, [10]

$$f(x) = \sin(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}}{(2n+1)!} x^{2n+1}$$ (3)

$$g(x) = \sqrt{e^{x+2}} = e \cdot e^{x/2} = \sum_{n=0}^{\infty} e \cdot \frac{1}{2^n \cdot n!} x^n$$ (4)

$$h(x) = \operatorname{sh}(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$$ (3)

———— o ————

9, [10]

$$f(x,y) = 2 + 3x^2y - xy^2 \;; \quad (x_0, y_0) = (1,2) \;; \quad \underline{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$f'_x(x,y) = 6xy - y^2 \;; \quad f'_x(1,2) = 12 - 4 = 8$$ (2) (1)

$$f'_y(x,y) = 3x^2 - 2xy \;; \quad f'_y(1,2) = 3 - 4 = -1$$ (2) (1)

$$|\underline{v}| = \sqrt{9+16} = 5 \;; \quad \underline{e} = \frac{\underline{v}}{|\underline{v}|} = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$$ (1)

$$\left.\frac{df}{d\underline{e}}\right|_{(x_0,y_0)} = \langle \operatorname{grad} f(x_0,y_0), \underline{e} \rangle = 8 \cdot \frac{3}{5} + (-1)\cdot\left(\frac{-4}{5}\right) = \frac{24+4}{5} =$$ (2)

$$= \frac{28}{5} = \underline{\underline{5 + \frac{3}{5}}}$$ (1)