Transzformációk

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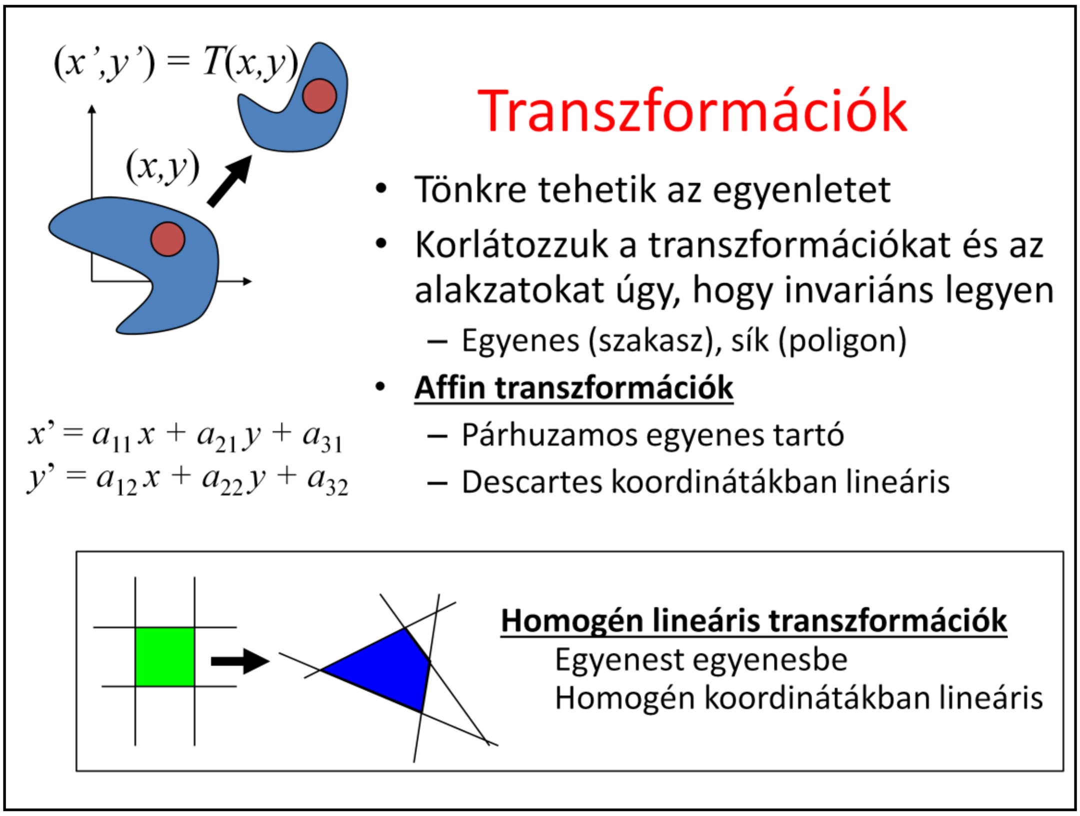
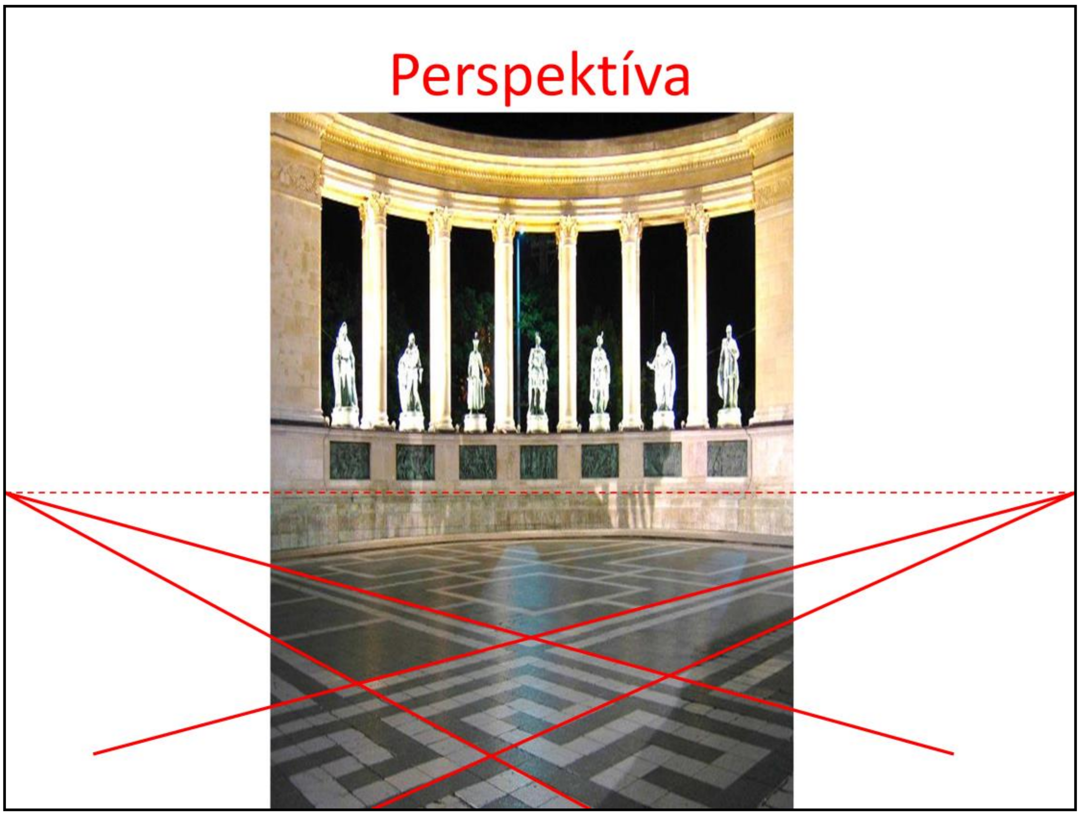


Geometric transformations assign a point to a point, so it is a point valued function of points. Geometric transformation may destroy the equation and the type of an object. Even simple scaling turns a sphere into an ellipsoid, so the equation, program, representation will change. To avoid this, we limit the allowed transformations and object types to those which guarantee that the object type is preserved. Linear elements, like points, line segments, and polygons may approximate any 0,1 or 2 dimensional object.

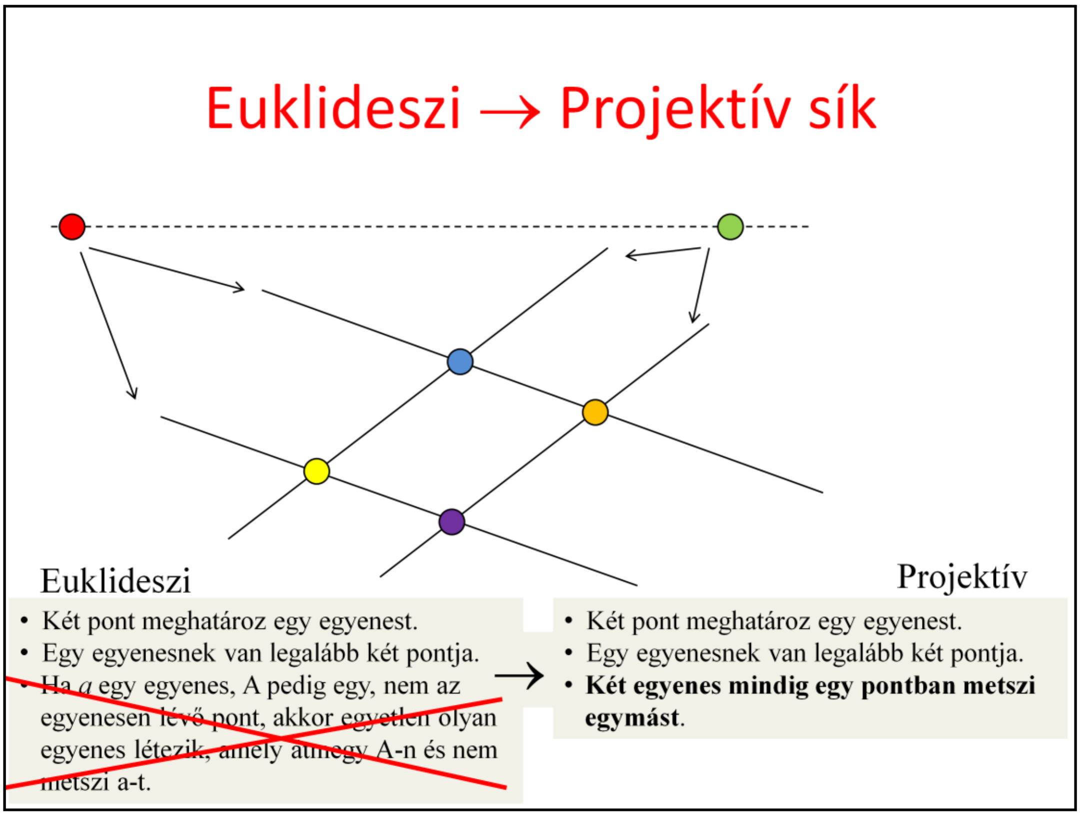
Affine transformations that can be expressed as linear functions of the Cartesian coordinates map lines to lines and also preserve parallel lines. This theorem can be proved by realizing that a line can have a linear equation and with linear equation only lines can be described. So, if a linear equation of a line is combined with the linear function of the transformation, we get a linear equation, which thus must be a line. If this transformation could make parallel lines intersecting or intersecting lines parallel, then this transformation would create a point out of nothing or would make a point disappear. A linear function is not able to do that..

Affine transformations are not the widest set of transformations preserving lines and polygons. The widest set is homogeneous linear transformations (homogeneous coordinates are multiplied by a matrix), which includes central projection as well. To find this wider set of transformations, we should understand that no transformation of the Euclidean plane can make two parallel lines intersecting, since that would create a point from nothing. The problem is the Euclidean geometry itself and its property that parallel lines do not intersect.

To consistently discuss how lines can be transformed to lines without keeping the parallelism, we should step out of the Euclidean geometry. The proper geometry is the projective geometry.



We can see the transformations of parallel lines to intersecting ones in every moment of our life. The phenomenon is called perspective.

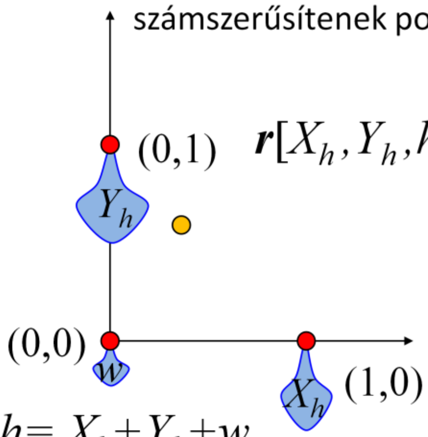


To establish projective geometry, the axioms need to change. The parallel axiom of the Euclidean geometry is deleted, and instead of this we postulate that „two lines intersect each other in exactly one point”. As a result, the Euclidean plane must be extended with ideal points. Each line is given one ideal point, assigning the same ideal point to two lines if and only if they are parallel. Ideal points will be on a line.

Homogén koordináták (2D)

$$\boxed{(x, y) \rightarrow [x, y, 1] \sim [x \cdot h, y \cdot h, h] = [X_h, Y_h, h]}$$

I. Homogén koordináták súlypont analógiával számszerűsítenek pontokat:



$$h = X_h + Y_h + w$$

$$\boldsymbol{r}[X_h, Y_h, h] = \frac{X_h \cdot (1,0) + Y_h \cdot (0,1) + w \cdot (0,0)}{h}$$

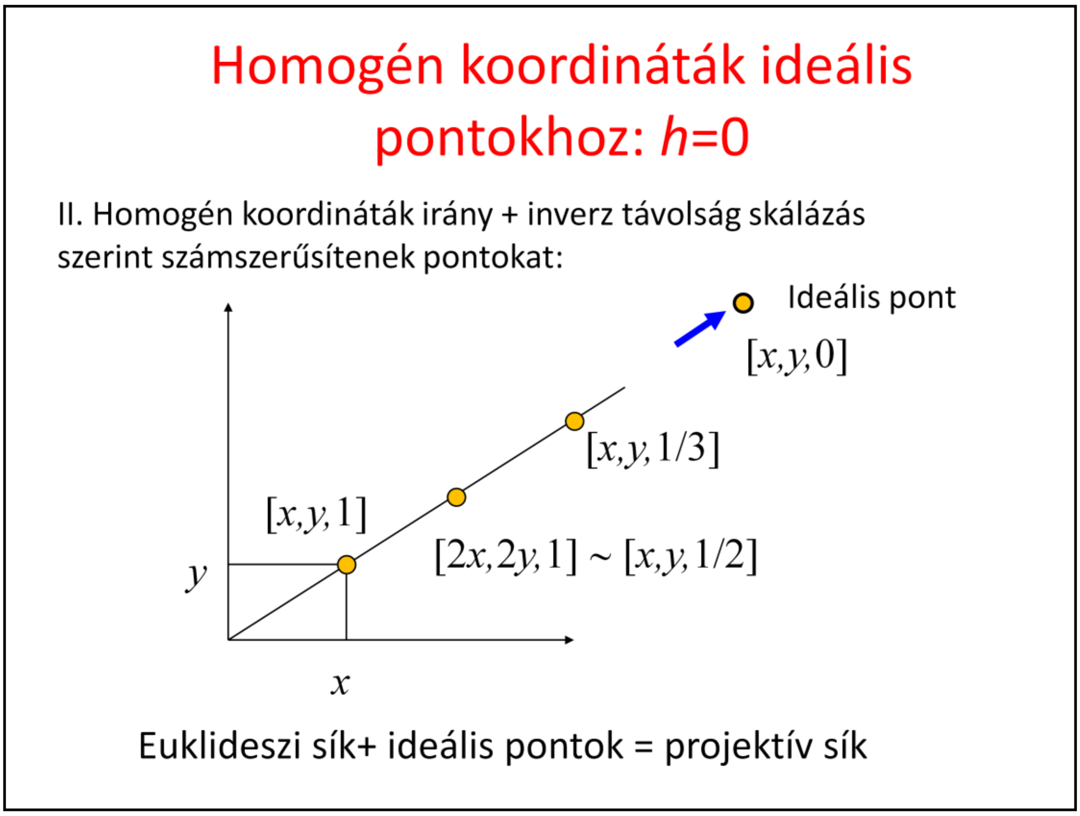
$$\boldsymbol{r}[X_h, Y_h, h] = \left(\frac{X_h}{h}, \frac{Y_h}{h}\right)$$

$$\boxed{x = \frac{X_h}{h} \qquad y = \frac{Y_h}{h}}$$

Homogeneous coordinates are defined by extending the Cartesian coordinates by an additional coordinate that is equal to 1, and multiplying all coordinates by an arbitrary non-zero scalar h. An intuitive interpretation of homogeneous coordinates in 2D is the following: we put weight Xh in point (1,0), weight Yh in point (0,1), and w=h-Xh-Yh in the origin. Value h is the total mass distributed. Three numbers Xh,Yh,h identify a point in 2D which is the center of mass of this mechanical coordinate system.

Based on the construction, it is obvious that any point of the Euclidean space, which can be given by Cartesian coordinates, can also be represented by homogeneous coordinates with non zero h. It is also true that any homogeneous coordinate triple where h is not zero, can also be given by Cartesian coordinates, which can be obtained by dividing the first two coordinates by the third.

So, if h is not zero, homogeneous coordinates can represent the same set of points as Cartesian coordinates.



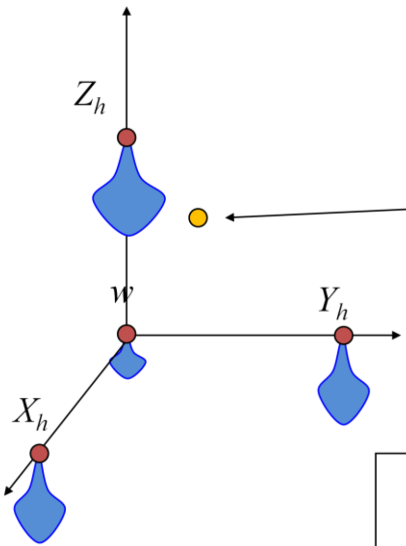
Homogeneous coordinates [Xh, Yh, h] can also be interpreted in the following way: (Xh, Yh) specify the direction of the point, and h is a scaling of the distance.

Let us consider a point of Cartesian coordinates x,y, which can be given in homogeneous coordinates as [x,y,1].

Now, let us consider another point that is in the same direction, but twice as far as (x,y). This farther point is (2x,2y) in Cartesian coordinates, [2x,2y,1] in homogeneous coordinates, or [x,y,1/2] in homogeneous coordinates. Similarly, the point that is also in the same direction but is f times farther away is [x,y,1/f]. So the interpretation of a homogeneous triplet is that the first two coordinates are Cartesian ones and show the direction, and the third coordinate is an inverse scaling of the distance. When f is infinity, so 1/f is zero, then we get [x,y,0], which is at the direction of (x,y), but at infinity.

With homogeneous coordinates we can express **ideal points**, i.e. points at infinity that are the intersections of parallel lines. Note that in Euclidean geometry parallel lines do not intersect. So, when we work with homogeneous coordinates instead of Cartesian ones, we describe the projective plane that contains the ideal points as well, and not the Euclidean plane.

Homogén koordináták (3D)



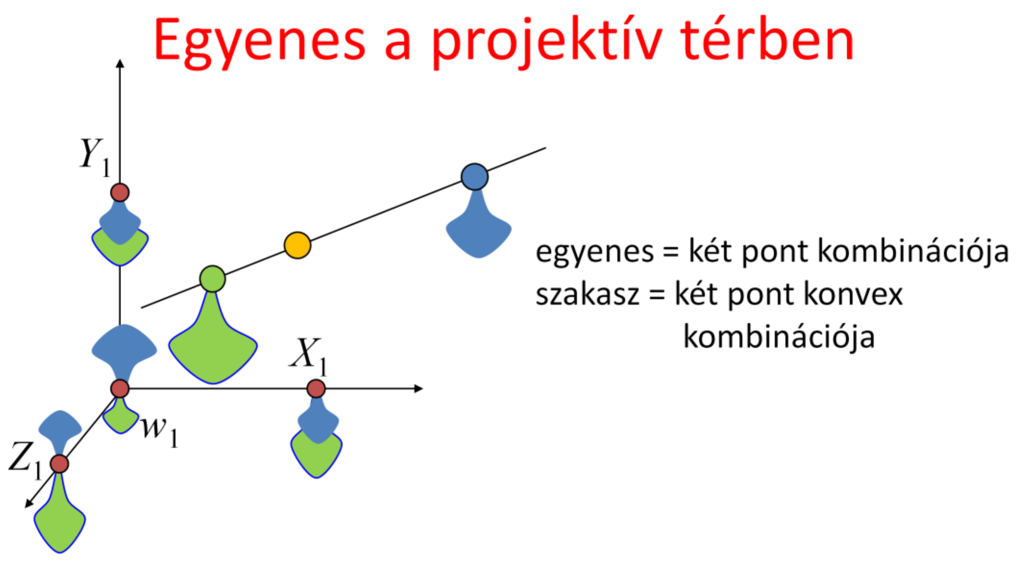
Teljes súly: $h = X_h + Y_h + Z_h + w$

Súlypont:

$\boxed{[X_h, Y_h, Z_h, h]}$ Nem lehet mind egyszerre zérus

$$\boxed{x = \frac{X_h}{h} \qquad y = \frac{Y_h}{h} \qquad z = \frac{Z_h}{h}}$$

3D points can also be represented with homogeneous coordinates, i.e. the 3D Cartesian space can also be extended to 3D projective space. The center of mass analogy puts weight Xh at reference point (1, 0, 0), weight Yh at (0,1,0), weight Zh at (0,0,1), and finally w = h–Xh-Yh-Zh at the origin. Using the definition of the center of mass, from a quadtuple of homogeneous coordinates, the corresponding Cartesian coordinate triplet can be obtained by homogeneous division (of course, only if h is not zero).



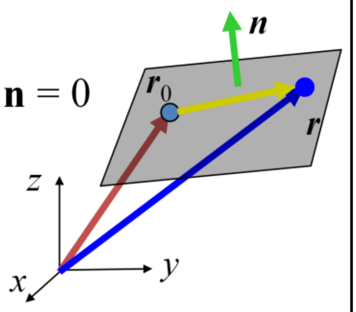
$$[X(t), Y(t), Z(t), h(t)] = [X_1, Y_1, Z_1, h_1] \cdot t + [X_2, Y_2, Z_2, h_2] \cdot (1\text{-}t)$$

We shall transform not only points but lines and planes as well, so we need the equations of lines and planes in homogeneous coordinates. We use the center of mass analogy. A point is specified by placing X1,Y1,Z1,h1-X1-Y1-Z1 weights at the ends of the basis vectors and the origin respectively, and another point is specified with X2,… weights. Both mechanical systems can be replaced by equivalent systems storing all weights in the center of mass. So when the two systems are combined, the final center of mass will be along a line between the two centers of masses. If we increase the weights of the first mechanical system proportionally scaling all weights, the location of the center of mass of the first system does not change, but it has larger total mass. So the center of mass of the combined system moves towards the first system along the line of the two centers of masses.

Thus, using this combination, we can obtain points on the line defined by the two centers of masses. If scaling is not negative, then we obtain the convex combination of the two points, which is a line segment. With allowing negative scaling, the total line can be specified.

Sík

$$(\mathbf{r} - \mathbf{r0}) \bullet \mathbf{n} = 0$$



Euklideszi tér, Descartes koordináták:

$$n_x x + n_y y + n_z z + d = 0$$

Euklideszi tér, homogén koordináták:

$$n_x X_h/h + n_y Y_h/h + n_z Z_h/h + d = 0 \qquad h \neq 0$$

Projektív tér:

$$\boxed{n_x X_h + n_y Y_h + n_z Z_h + d\,h = 0} \qquad \cancel{h \neq 0}$$

$$[X_h, Y_h, Z_h, h] \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \\ d \end{bmatrix} = 0$$

In Euclidean geometry, using Cartesian coordinates, the plane is a linear equation of the coordinates. To find the plane in projective space, the points at infinity are added to this plane. First, Cartesian coordinates are replaced by homogeneous ones, assuming that h is not zero (it is forbidden to divide by zero). Then, both sides are multiplied with h. In this new equation we do not divide by h, so we can ignore the ”h is not zero” requirement. This corresponds to adding ideal points to the plane.

The projective plane is thus a homogeneous linear equation of homogeneous coordinates. We can also express it as a dot product of two 4D vectors, one describes the point, the other the parameters of the plane.

Homogén lineáris transzformációk

- **<u>Homogén koordinátavektor szorzása mátrixszal</u>**
- 2D transzformáció 3x3 mátrix

$$[X_h', Y_h', h'] = [X_h, Y_h, h] \cdot \mathbf{T}_{3x3}$$

- 3D transzformáció 4x4 mátrix

$$[X_h', Y_h', Z_h', h'] = [X_h, Y_h, Z_h, h] \cdot \mathbf{T}_{4x4}$$

- Transzformációk konkatenációja: Asszociatív

$$\begin{aligned}[X_h', Y_h', Z_h', h'] &= (\ldots([X_h, Y_h, Z_h, h] \cdot \boldsymbol{T}_1) \cdot \boldsymbol{T}_2) \ldots \boldsymbol{T}_n) = \\ &= [X_h, Y_h, Z_h, h] \cdot (\boldsymbol{T}_1 \cdot \boldsymbol{T}_2 \cdot \ldots \cdot \boldsymbol{T}_n) = \\ &= [X_h, Y_h, Z_h, h] \cdot \boldsymbol{T}\end{aligned}$$

Homogeneous linear transformations are the multiplications of the vector of homogeneous coordinates by a matrix. The vector can be a row vector when it is on the left side of the matrix. On the other hand, the vector can also be a column vector, and stands on the right side. The two approaches are similar, just the matrix should be transposed accordingly. We shall prefer the case when the vector is a row vector, because it is more intuitive when multiple transformations are executed on after the other.

A 2D point is described by 3 homogeneous coordinates, thus the transformation matrix is of 3x3 size.

For 3D points, the matrix has 4x4 elements.

In practice we execute not only a single transformation, but a sequence of transformations. This can be imagined as transforming the point with T1, then the result by T2, etc. However, as matrix multiplication is associative, i.e. parentheses can be regrouped, we obtain the same result if we multiply the point with the product of concatenation of the transformation matrices. Any sequence of transformations can be expressed as a single matrix multiplication. If we consider points as row vectors, then the order of transformation matrices will correspond to the order of their execution.

Affin transzformációk

- Ha az utolsó oszlop $[0,0,1]^T$ vagy $[0,0,0,1]^T$
- Descartes koordinátákra lineáris
- Párhuzamos egyenestartó

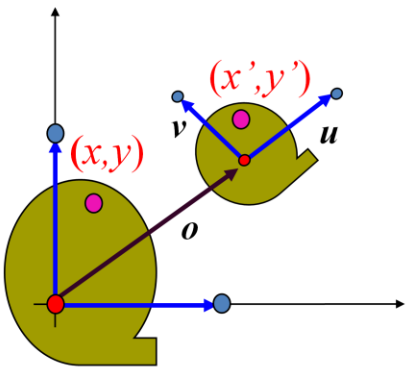
$$x' = a_{11}x + a_{21}y + a_{31}$$
$$y' = a_{12}x + a_{22}y + a_{32}$$

$$[x', y', 1] = [x, y, 1]\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix}$$

If the last column of the matrix is 0,0,1 in 2D and 0,0,0,1 in 3D, then the transformation is affine, i.e. it maps lines to lines and preserves parallel lines. From another point of view, the new Cartesian coordinates are linear functions of the original Cartesian coordinates.

Such transformation matrices do not modify the last homogeneous coordinate h.

Affin transzformációs mátrix sorai



$$(x', y') = \boldsymbol{o} + x\boldsymbol{u} + y\boldsymbol{v}$$

$$\cdot\begin{bmatrix} u_x & u_y & 0 \\ v_x & v_y & 0 \\ o_x & o_y & 1 \end{bmatrix}$$

$$\begin{array}{ll} [0, 0, 1] & [o_x \quad o_y \quad 1\,] \\ [1, 0, 1] & [o_x+u_x \; o_y+u_y \; 1\,] \\ [0, 1, 1] & [o_x+v_x \; o_y+v_y \; 1\,] \end{array}$$

In case of affine transformations, the third column is [0, 0, 1] and the row vectors of the remaining part of the matrix have important meaning. They describe what happens with basis vector i, j, and the origin if the transformation is executed.

Homogén lineáris transzformációk tulajdonságai

- Egyenest egyenesbe, konvex kombinációkat konvex kombinációkba képeznek le

Példa: egyenest egyenesbe

$$[X(t), Y(t), Z(t), h(t)] = [X_1, Y_1, Z_1, h_1] \cdot t + [X_2, Y_2, Z_2, h_2] \cdot (1-t)$$

$$P(t) = P_1 \cdot t + P_2 \cdot (1-t) \quad // \cdot \mathbf{T}$$

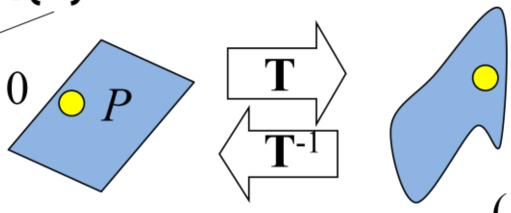
$$P^*(t) = P(t) \cdot \mathbf{T} = (P_1 \cdot \mathbf{T}) \cdot t + (P_2 \cdot \mathbf{T}) \cdot (1-t)$$

Homogeneous linear transformations are matrix multiplications of 4 element vectors in 3D and 3 element vectors in 2D. Such linear operations preserve linear computations, so a line is transformed to a line or to a point if the line degenerates, which never happens if T is invertible.

Invertálható homogén lineáris transzformációk: síkot síkba

$$[X_h, Y_h, Z_h, h] \begin{bmatrix} n_x \\ n_y \\ n_z \\ d \end{bmatrix} = 0$$

$$P \cdot N^T = 0$$



$$P^* = P \cdot \mathbf{T}$$

$$(P^* \cdot \mathbf{T}^{-1}) \cdot N^T = 0$$

$$P^* \cdot (\mathbf{T}^{-1} \cdot N^T) = 0$$

$$P^* \cdot N^{*T} = 0$$

$$\boxed{N^* = N \cdot (\mathbf{T}^{-1})^T}$$ Inverse transpose

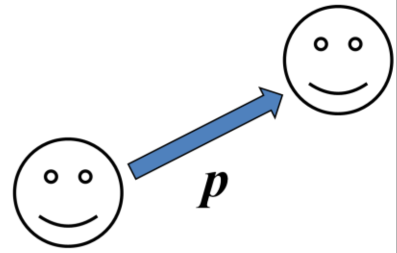
Invertible homogeneous transformations map planes to planes. If the transformation is not invertible, it may happen that the resulting plane degenerates to a line or to a point.

A plane is a collection of points P that satisfy the plane equation. Multiplying every point P by matrix T, we get a collection of points P*. To find an equation for P*, we transform P* back to get P since we know that P satisfies the original equation.

As matrix multiplication is associative, we express a similar equation for the transformed points as well, so they are also on a plane. We can even determine the parameters of the plane (e.g. normal vector). If the parameters are a column vector, the parameters of the original plane must be left-multiplied with the inverse of the transformation matrix.

Eltolás

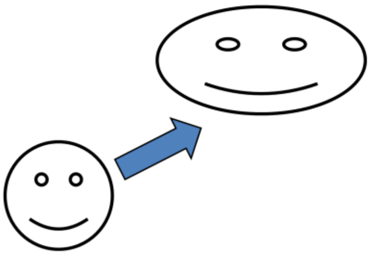
$$(x', y', z') = (x+p_x, y+p_y, z+p_z)$$



$$[x', y', z', 1] = [x, y, z, 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p_x & p_y & p_z & 1 \end{bmatrix}$$

The first elementary transformation considered is the 3D translation. This transformation computes the sum of the Cartesian coordinates of the point and of the translation vector p. This operation can be represented by a homogeneous transformation matrix, where the diagonal elements are 1, the last row contains the translation vector and all other elements are zero.

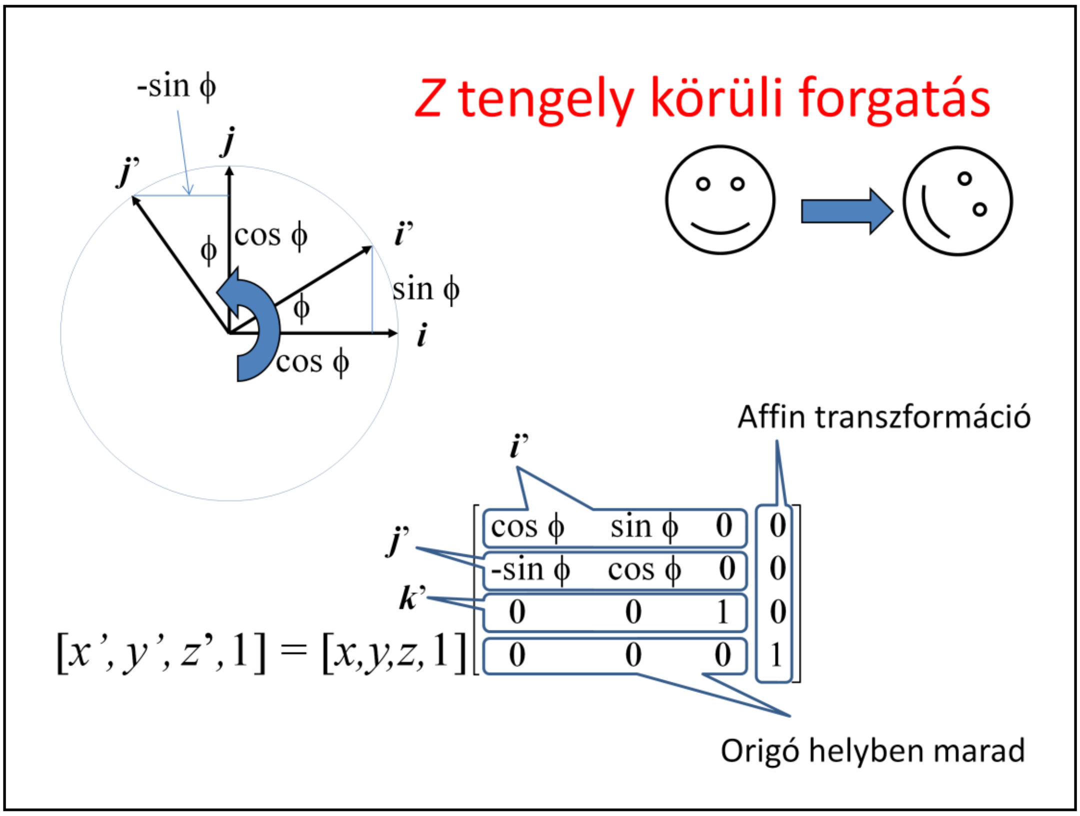
Skálázás



$$x' = S_x \cdot x, \quad y' = S_y \cdot y, \quad z' = S_z \cdot z$$

$$[x', y', z', 1] = [x, y, z, 1] \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The second transformation is scaling along the coordinate axes. This scales x coordinates by Sx, y coordinates by Sy and z coordinates by Sz. Scaling is a diagonal homogeneous linear transformaiton, including the scaling factors and 1 in the diagonal.



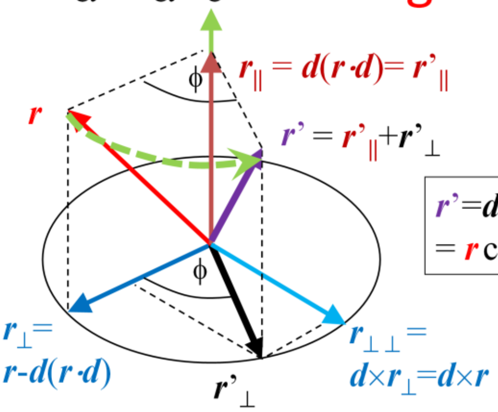
Rotation, for example around axis z, is a congruence transformation, thus it surely belongs to the category of homogenous linear transformations.

If we rotate around axis z, coordinate z is left unchanged and x, y are modified. Let us express x, y with polar coordinates r, alpha. Rotation does not modify r, but the polar angle is increased by the rotation angle phi.

Using trigonometric identities, we can express the transformed point's x', y' coordinates, which indeed can be realized by a matrix multiplication.

Általános $\boldsymbol{d}$ tengely körüli forgatás: Rodrigues formula

$\boldsymbol{d}$: forgatási tengely, legyen egységvektor



$$\begin{array}{l}
\boxed{\begin{aligned}\boldsymbol{r}' &= \underbrace{\boldsymbol{d}(\boldsymbol{r}\cdot\boldsymbol{d})}_{\boldsymbol{r}'_{\|}} + \underbrace{(\boldsymbol{r}-\boldsymbol{d}(\boldsymbol{r}\cdot\boldsymbol{d}))}_{\boldsymbol{r}_{\perp}}\cos(\phi) + \underbrace{\boldsymbol{d}\times\boldsymbol{r}}_{\boldsymbol{r}_{\perp\perp}}\sin(\phi)\\ &= \boldsymbol{r}\cos(\phi) + \boldsymbol{d}(\boldsymbol{r}\cdot\boldsymbol{d})(1-\underbrace{\cos(\phi)}_{C}) + \boldsymbol{d}\times\boldsymbol{r}\underbrace{\sin(\phi)}_{S}\end{aligned}}
\end{array}$$

<u>Mátrix sorai: hova kerül az origó és az $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$?</u>

$(0,0,0) \rightarrow (0,0,0)$

$(1,0,0) \rightarrow (1,0,0)C + (\boldsymbol{d}_x,\boldsymbol{d}_y,\boldsymbol{d}_z)\boldsymbol{d}_x(1-C) + (\boldsymbol{d}_x,\boldsymbol{d}_y,\boldsymbol{d}_z)\times(1,0,0)S =$

$\quad (C+\boldsymbol{d}_x^2(1-C),\ \boldsymbol{d}_y\boldsymbol{d}_x(1-C) + \boldsymbol{d}_z S,\ \boldsymbol{d}_z\boldsymbol{d}_x(1-C) - \boldsymbol{d}_y S)$

Let us find the matrix rotating around a line crossing the origin and of direction $\boldsymbol{w}$. For the sake of notational simplicity, w is assumed to have unit length.

Let us decompose the vector to be rotated, r, into a component that is parallel with $\boldsymbol{w}$, $\boldsymbol{r}_{\|} = \boldsymbol{w}(\boldsymbol{r}\cdot\boldsymbol{w})$, and a vector that is perpendicular to it, $\boldsymbol{r}_{\perp} = \boldsymbol{r} - \boldsymbol{w}(\boldsymbol{r}\cdot\boldsymbol{w})$. The parallel vector is not changed by the rotation. The perpendicular component remains in the plane that is perpendicular to $\boldsymbol{w}$.

The rotated perpendicular vector is expressed as a linear combination of $\boldsymbol{r}_{\perp}$, and a vector that is in the same plane and is perpendicular to $\boldsymbol{r}_{\perp}$. This vector is $\boldsymbol{w}\times\boldsymbol{r}_{\perp} = \boldsymbol{w}\times\boldsymbol{r}$. If $\boldsymbol{r}_{\perp}$ is rotated by angle alpha, then it will be $\boldsymbol{r}'_{\perp}\cos(\alpha) + \boldsymbol{w}\times\boldsymbol{r}_{\perp}\sin(\alpha)$.

Making the substitutions, we get the Rodrigues formula. How do we get a matrix? We should evaluate this formula or (1, 0, 0) and get the first 3 components of the first row vector of the matrix. The other two rows are obtained similarly.

Kvaterniók újratöltve

Rodriguez: $\boldsymbol{r}' = \boldsymbol{r}\cos(\phi) + \boldsymbol{d}(\boldsymbol{r}\cdot\boldsymbol{d})(1-\cos(\phi)) + \boldsymbol{d}\times\boldsymbol{r}\sin(\phi)$

Kvaternió: $\boldsymbol{q} = [\cos(\phi/2), \boldsymbol{d}\sin(\phi/2)]$, $\boldsymbol{q}\cdot[0,\boldsymbol{r}]\cdot\boldsymbol{q}^{-1} = [0,\boldsymbol{r}']$

$[s_1, \boldsymbol{d}_1] \cdot [s_2, \boldsymbol{d}_2] = [s_1 s_2 - \boldsymbol{d}_1\cdot\boldsymbol{d}_2 , s_1\boldsymbol{d}_2 + s_2\boldsymbol{d}_1 + \boldsymbol{d}_1\times\boldsymbol{d}_2]$

<u>Bizonyítás $\boldsymbol{d}$ merőleges $\boldsymbol{r}$ esetre (párhuzamos HF):</u>

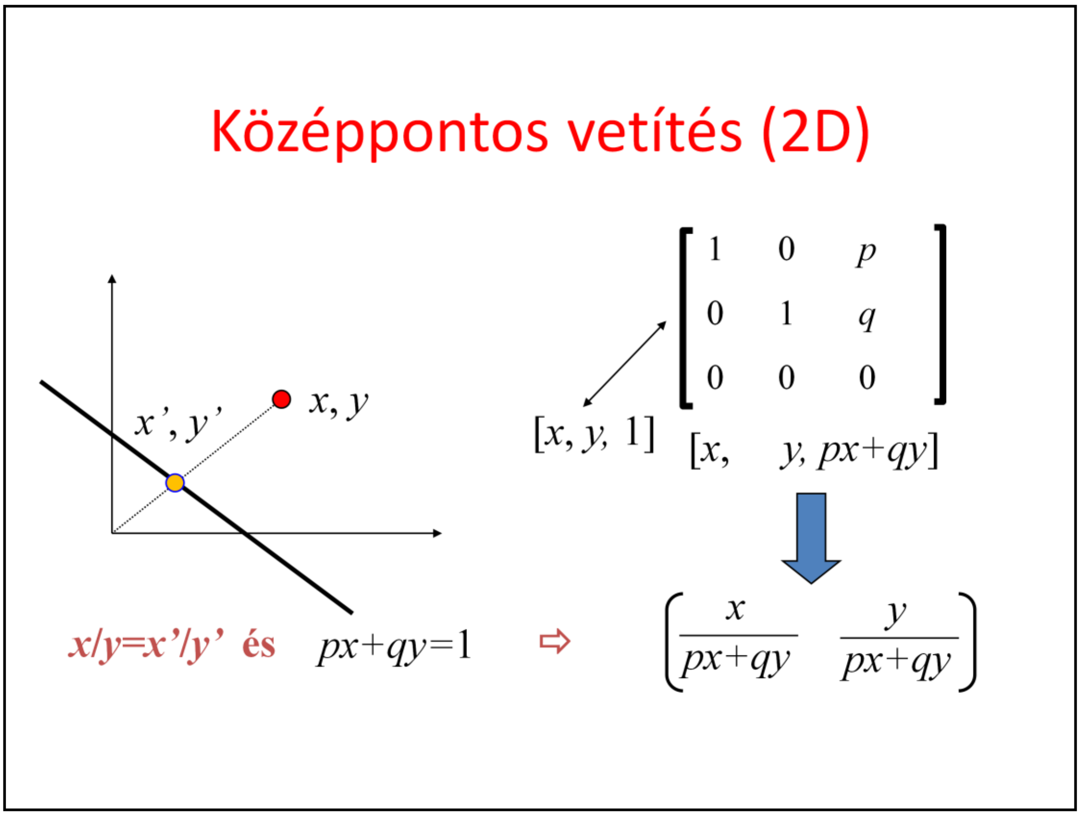
Rodriguez: $\boldsymbol{r}' = \boldsymbol{r}\cos(\phi) + \boldsymbol{d}\times\boldsymbol{r}\sin(\phi)$

Kvaternió:

$[\cos(\phi/2), \boldsymbol{d}\sin(\phi/2)]\cdot[0,\boldsymbol{r}] = [0, \boldsymbol{r}\cos(\phi/2) + \boldsymbol{d}\times\boldsymbol{r}\sin(\phi/2)] = [0,\boldsymbol{r}^*]$

$[0,\boldsymbol{r}^*]\cdot[\cos(\phi/2), -\boldsymbol{d}\sin(\phi/2)] = [0, \boldsymbol{r}^*\cos(\phi/2) - \boldsymbol{r}^*\times\boldsymbol{d}\sin(\phi/2)]$

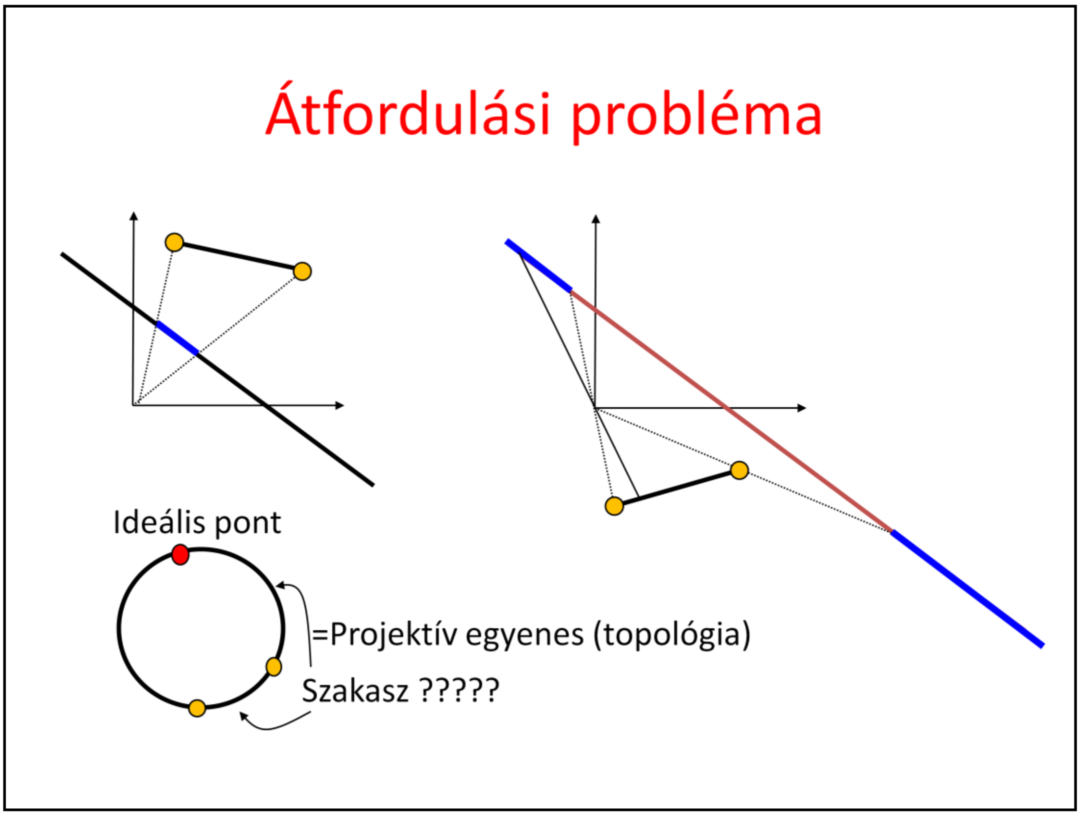
Knowing the algebraic formula for the rotation around an arbitrary axis, we can prove that the learnt method of quaternions indeed provides the same effect. Here we show the proof for the case when the rotation axis is perpendicular to the position vector of the rotated point. The parallel case is left for you.



So far, we discussed affine transformations by first introducing them and then developing a matrix for each of them. Let us now reverse the direction of this process and consider a 3x3 matrix, i.e. a transformation in the 2D plane and let us determine what this transformation does. To make it more exciting, the third column is not 0, 0, 1, so it is probably a non-affine transformation. Executing the vector-matrix multiplication, we can obtain the transformation of point (x,y) in homogeneous and also in Cartesian coordinates. Note that the new Cartesian coordinates are non-linear functions of the original Cartesian coordinates, so this transformation is not affine.

What does this transformations? It is a central projection onto a line of equation px+qy=1 assuming the origin as the center of the projection.

With homogeneous linear transformations we can express even non affine transformations but can still be sure that this transformation maps lines to lines, line segments to line segments, etc.



Let us execute this transformation for line segments. We are happy because it is enough to transform the two endpoints and the transformed pair of points can be connected by a line segment according to the properties of homogeneous linear transformations. For the first example, this is indeed true. However, for the second example, the transformation is seemingly not a line segment but its complement on the line, i.e. two half lines.

This is just a virtual contradiction. These two half lines also form a line segment in projection plane. The ideal point at the ”end” of the line glues the two ends together. The conclusion is that we should be careful since two points on a line can define two line segments that complement each other, similarly as two points on a circle can define two complementing arcs (a line in projective plane is topologically equivalent to a circle, we can go around it).

Ellenőrző kérdések

- Bizonyítsa be, hogy ha a transzformált *x′, y′* koordináták az eredeti *x, y*-nak lineáris függvényei, akkor a transzformáció egyenest egyenesbe képez le és a párhuzamos egyeneseket megtartja!
- Lehet-e egy affin transzformációnak olyan mátrixa, ahol az utolsó oszlop nem [0, 0, 0, 1]?
- Írja fel az adott irányú, origón átmenő tengely körül alfa szöggel forgató transzformáció mátrixát!
- Írja fel a vektoriális szorzást mátrixművelettel?
- Írja fel egy síkra merőlegesen vetítő, illetve centrálisan vetítő transzformációk mátrixait!
- Hogyan oldható fel az átfordulási probléma?
- Milyen alakzat az összes ideális pontot tartalmazó halmaz?
- Írja fel egy parabola egyenletét a projektív síkon!
- Határozza meg két párhuzamos egyenes metszéspontjának (homogén) koordinátáit a projektív síkon!
- Adjon meg transzformációt, amely egy adott háromszöget egy másik adott háromszögbe képez le!
- *Adjon meg transzformációt, amely egy konvex négyszöget egy konvex négyszögbe képez le! Mi történik, ha a négyszög nem konvex?*