MINTAMEGOLDÁS (α) $\boxed{-1-}$ 2013. 06. 13.

1, Homogén: (H) $y'(x) + f(x)\, y(x) = 0$

a, Inhomogén: (I) $y'(x) + f(x)\cdot y(x) = g(x)$

$\boxed{6}$

T.: Ha y_1 és y_2 megoldása (I)-nek, akkor $y_1 - y_2$ megoldása (H)-nak. (2)

B.:
$$\begin{aligned} & y_1' + f\, y_1 = g \\ \ominus\ & y_2' + f\, y_2 = g \end{aligned}$$

$$\underbrace{y_1' - y_2'}_{(y_1 - y_2)'} + \underbrace{f y_1 - f y_2}_{f(y_1 - y_2)} = 0$$ (4)

$$(y_1 - y_2)' + f(y_1 - y_2) = 0 \quad ✓$$

b, $y' + y\, \mathrm{tg}\, x = \sin(2x)$ $y(\pi/4) = 1.$

$x \in \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$

$\boxed{10}$

(5)

(H): $y' = \frac{dy}{dx} = -y\, \mathrm{tg}\, x$

$$\int \frac{dy}{y} = -\int \frac{\sin x}{\cos x}\, dx \implies \ln|y| = \ln|\cos x| + C$$

$$\implies \underline{y_{H,ált}(x) = K \cdot \cos x}$$

(4)

Variálás: $y_{I,p}(x) = K(x) \cos x \implies y_{I,p}'(x) = K' \cos x - K \sin x$

Beírva: $K' \cos x - \cancel{K \sin x} + \cancel{K \cos x\, \mathrm{tg}\, x} = \underbrace{\sin 2x}_{2 \sin x \cos x}$

$$K'(x) = 2 \sin x \implies K(x) = -2 \cos x$$

$$\implies y_{I,ált}(x) = y_{H,ált}(x) + y_{I,p}(x) = \underline{K \cos x - 2 \cos^2 x}$$

(1)

Kezd. felt.: $1 = \frac{K}{\sqrt{2}} - 2/2 \implies K = 2\sqrt{2}$

$$\implies \underline{y_{kezd.}(x) = 2\sqrt{2} \cos x - 2 \cos^2 x}$$

2, a, $3x^2 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$; $e^{4x} \Rightarrow \lambda_4 = 4$; $\lambda^3 \cdot (\lambda - 4) = \lambda^4 - 4\lambda^3$ (2)

(4) $y^{(4)} - 4y''' = 0$ (1)

$y_{ált}(x) = Ax^2 + Bx + C + De^{4x}$ (1)

b, $\sin(2x) \Rightarrow \lambda_{1,2} = \pm 2i$; $2e^{-3x} \Rightarrow \lambda_3 = -3$; $\overbrace{(\lambda + 2i)(\lambda - 2i)}^{\lambda^2 + 4}(\lambda + 3) = \lambda^3 + 3\lambda^2 + 4\lambda + 12$ (2)

(4) $y''' + 3y'' + 4y' + 12y = 0$ (1)

$y_{ált} = A\sin(2x) + B\cos(2x) + Ce^{-3x}$ (1)

c, (6) $ch(2x)e^{3x} = \dfrac{e^{2x} + e^{-2x}}{2} e^{3x} = \dfrac{1}{2}e^{5x} + \dfrac{1}{2}e^{x}$ (2)

$\Rightarrow \lambda_1 = 5,\ \lambda_2 = 1$; $(\lambda - 5)(\lambda - 1) = \lambda^2 - 6\lambda + 5$

$y'' - 6y' + 5y = 0$; $y_{ált} = Ae^{5x} + Be^{x}$ (4)

3, a, $\boxed{7}$ Binomiális sor:

$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$, ha $|x| < R = 1$, $\alpha \in \mathbb{R}$ (2)

$\binom{\alpha}{k} = \dfrac{\alpha \cdot (\alpha - 1) \cdot \ldots \cdot (\alpha - k + 1)}{k!}$

T.: A binom. sor konvergenciasugara $R = 1$ (1)

B.: Hányados kritériummal:

(4) $\left| \dfrac{a_{k+1}}{a_k} \right| = \left| \dfrac{\binom{\alpha}{k+1}}{\binom{\alpha}{k}} \right| = \left| \dfrac{\alpha - k}{k+1} \right| = \left| \dfrac{\frac{\alpha}{k} - 1}{1 + \frac{1}{k}} \right| \xrightarrow{k \to \infty} |-1| = 1 \Rightarrow R = 1$

b,
$\boxed{9}$ $f(x) = \dfrac{1}{\sqrt[3]{27+9x^2}} = \dfrac{1}{3}\,\dfrac{1}{\sqrt[3]{1+\frac{x^2}{3}}} = \dfrac{1}{3}\left(1+\dfrac{x^2}{3}\right)^{-1/3} =$

$$= \frac{1}{3}\sum_{k=0}^{\infty}\binom{-1/3}{k}\left(\frac{x^2}{3}\right)^k = \sum_{k=0}^{\infty}\binom{-1/3}{k}\cdot\frac{x^{2k}}{3^{k+1}} \qquad (4)$$

$$\left|\frac{x^2}{3}\right| < 1 \implies |x| < \sqrt{3} = R \qquad (2)$$

$$f^{(6)}(0) = 6!\cdot a_6 \underset{\substack{\uparrow \\ k=3}}{=} 6!\cdot\binom{-1/3}{3}\cdot\frac{1}{3^4} = \frac{6!}{3^4}\cdot\frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{3!} \qquad (3)$$

4,
$\boxed{14}$ $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & \text{ha } (x,y) \neq (0,0) \\ 0, & \text{ha } (x,y) = (0,0) \end{cases}$

(5) a, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{r\to 0+} \dfrac{r^2 \sin\varphi\cos\varphi}{r^2} = \sin\varphi\cos\varphi = \not\exists$ függ φ-től

$\implies$ f-nek nincs határértéke $(0,0)$-ban $\implies$ f nem folyt. $(0,0)$-ban

(4) b, Ha $(x,y) \neq (0,0)$

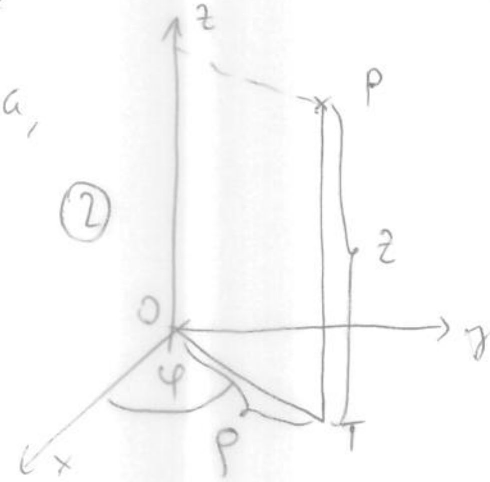
$f'_x = \dfrac{y(x^2+y^2) - xy(2x)}{(x^2+y^2)^2} = \dfrac{y^3 - x^2y}{(x^2+y^2)^2}$; $f'_y = \dfrac{x^3 - xy^2}{(x^2+y^2)^2}$

(3) c, Ha $(x,y) = (0,0)$

$f'_x(0,0) = \lim\limits_{x\to 0} \dfrac{f(x,0)-f(0,0)}{x} = \lim\limits_{x\to 0}\dfrac{0-0}{x} = 0 = f'_y(0,0)$

(2) d, Nem létezik a gradiens az origóban, mert a függvény ott nem folytonos.

5*, a, [6]



(2)

Henger koord.: (ρ, φ, z)

P vetülete az (x, y) síkra: T

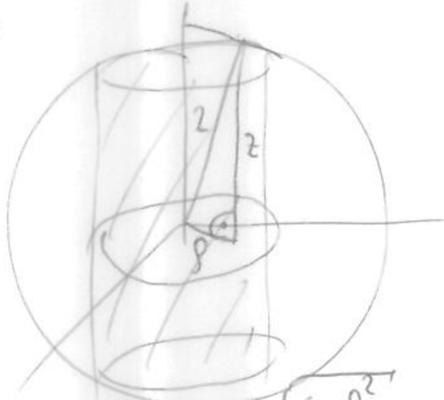
$\rho = |\overline{OT}|$; φ = poz. x és OT szöge;

$z = PT$

$x = \rho\cos\varphi$; $y = \rho\sin\varphi$, $z = z$ ← (2)

(2) $$\mathcal{J} = \begin{vmatrix} x'_\rho & x'_\varphi & x'_z \\ y'_\rho & y'_\varphi & y'_z \\ z'_\rho & z'_\varphi & z'_z \end{vmatrix} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho(\cos^2\varphi + \sin^2\varphi) = \underline{\underline{\rho}}$$

b, [8]



Hengerkoordinátákkal számolunk:

$0 \le \rho \le 1$ (1)

$0 \le \varphi \le 2\pi$ (1)

$-\sqrt{4-\rho^2} \le z \le \sqrt{4-\rho^2}$ (2)

jac. (2) $f' \cdot f^{1/2}$ alak

$$V = \int_{\rho=0}^{1} \int_{\varphi=0}^{2\pi} \int_{z=-\sqrt{4-\rho^2}}^{\sqrt{4-\rho^2}} 1 \cdot \rho \, dz \, d\varphi \, d\rho = 2\pi \cdot \int_{\rho=0}^{1} 2\rho\sqrt{4-\rho^2} \, d\rho =$$

(2)

$$= -2\pi \left[(4-\rho^2)^{3/2} \cdot \frac{2}{3} \right]_0^1 = -\frac{4\pi}{3}\left(3^{3/2} - 4^{3/2}\right) = \underline{\underline{\frac{4\pi}{3}\left(8 - 3^{3/2}\right)}}$$

6*, a, [4] $u(x,y) = x^3 - 3xy^2 - 2xy$; $u'_x = 3x^2 - 3y^2 - 2y$ (1)

$\Delta u = u''_{xx} + u''_{yy} =$ | $u'_y = -6xy - 2x$ (1)

$= 6x + (-6x) = 0$ ✓ (2)

(– 5 –)

b, [8] C-R: $\left.\begin{aligned} u'_x &= +v'_y = 3x^2 - 3y^2 - 2y \\ u'_y &= -v'_x \end{aligned}\right\}$ ②

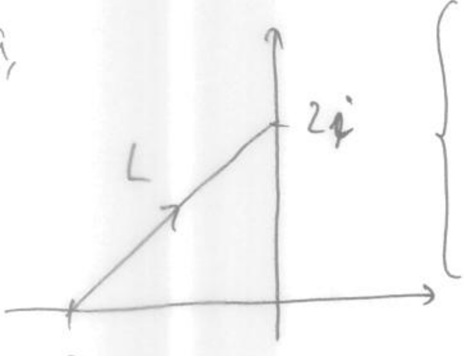
$v_y = \int v'_y \, dy = \int (3x^2 - 3y^2 - 2y)\, dy = 3x^2 y - y^3 - y^2 + C(x)$ ②

$\left.\begin{aligned} v'_x = 6xy + C'(x) \overset{!}{=} -u'_y = 6xy + 2x &\Rightarrow C'(x) = 2x \\ &\Rightarrow C(x) = x^2 + K \end{aligned}\right\}$ ②

② $\left\{\begin{aligned} &\text{Tehát } \underline{v(x,y) = 3x^2y - y^3 - y^2 + x^2 + K}, \text{ és} \\ &\underline{f(z) = u(x,y) + i\,v(x,y)} = (x^3 - 3xy^2 - 2xy) + i(3x^2y - y^3 - y^2 + x^2 + K) \end{aligned}\right.$

$K \in \mathbb{R}$.

7* a, [6]



$\left\{\begin{aligned} &z(t) = t + (t+2)i \; ; \quad t = -2 .. 0 \\ &z\bar{z} = (t + (t+2)i)\cdot(t - (t+2)i) = \\ &\quad = t^2 + (t+2)^2 = 2t^2 + 4t + 4 \\ &\dot{z}(t) = 1 + i \end{aligned}\right.$

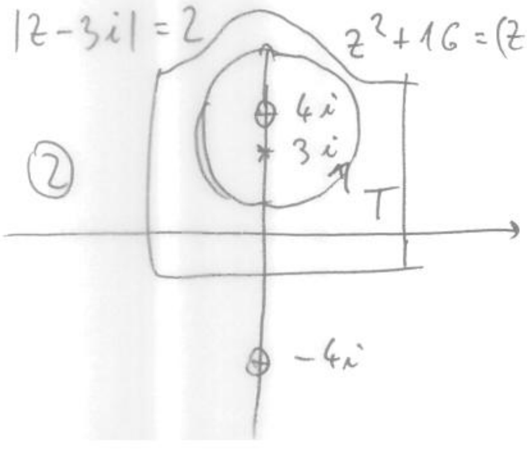
$$\int_L z\bar{z}\, dz = \int_{t=-2}^{0} \underbrace{(2t^2+4t+4)}_{z\bar{z}} \cdot \underbrace{(1+i)}_{\dot{z}}\, dt = (1+i)\left[\frac{2t^3}{3} + 2t^2 + 4t\right]_{-2}^{0} =$$ ④

$$= (1+i)\left(\frac{16}{3} \cancel{-8} \cancel{+8}\right) = \underline{\underline{\frac{16}{3} + \frac{16}{3} i}}$$ ②

b, [8]

$$\oint_{|z-3i|=2} \frac{\ln z}{z^2+16}\, dz = \oint_{|z-3i|=2} \frac{\left(\frac{\ln z}{z+4i}\right)}{z-4i}\, dz \overset{\text{Ch. I. int.}}{=} 2\pi i \left.\frac{\ln z}{z+4i}\right|_{z=4i} =$$ ② reg. T-n ②

$z^2 + 16 = (z-4i)(z+4i)$



$$= 2\pi i \, \frac{\ln(4i)}{8i} = \underline{\underline{\frac{\pi}{4} \cdot \left(\ln 4 + i\,\frac{\pi}{2}\right)}}$$ ②

Pótfeladatok:

8. a) $3a_{n+1} = -5a_n + 2a_{n-1}$

[6] $3q^2 = -5q + 2 \Longrightarrow 3q^2 + 5q - 2 = 0$ (2)

$$q_{1,2} = \frac{-5 \pm \sqrt{25+24}}{6} = \frac{-5 \pm 7}{6} = \begin{cases} -2 \\ +\frac{1}{3} \end{cases}$$ (2)

(2)

$$a_n = A(-2)^n + B\left(\frac{1}{3}\right)^n$$

[4] b) Akkor, ha $A = 0$. Tehát $a_1 = \frac{1}{3}$, $a_n = \left(\frac{1}{3}\right)^n$

9. a) $f(x) = \sin x \cdot \cos x = \frac{1}{2}\sin(2x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(2x)^{2k+1}$

[5]

b) $\operatorname{sh} x = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x} = \frac{e}{2}e^{(x-1)} - \frac{1}{2e}\cdot e^{-(x-1)} =$

[5]

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{e}{2} - \frac{(-1)^k}{2e}\right)(x-1)^k$$

$$\left(e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!} \right)$$