Advanced Performance Modeling and Analysis

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July 3, 2019.

Outline

- Non-Markovian queues
  - M/G/1, G/M/m, G/G/1 queues
- Matrix geometric methods
  - Phase type distributions
  - Markov arrival process
  - Quasi birth-death processes
  - Solution methods
- Fluid queues
  - Types (infinite/finite, first/second order, homogeneous/inhomogeneous)
  - Spectral and diff. eq. based solutions
  - Matrix analytic solution

Syllabus: Probability

CDF: $F(t) = Pr(X \leq t)$,

PDF: $f(t) = \frac{d}{dt}F(t)$,

Hazard rate – intensity: $\lambda(t) = \frac{f(t)}{1 - F(t)}$,

Expectation: $E(G(X)) = \int_t G(t)\ dF(t)$

Moments: $E(X^n) = \int_t t^n\ dF(t)$

Laplace transform: $F^{\sim}(s) = E(e^{-sX}) = \int_t e^{-st}\ dF(t)$

Z transform: $N(z) = E(z^N) = \sum_i p_i z^i$

Syllabus: Properties of transforms

The distribution of a r.v. is uniquely defined by

- Distribution function (or PDF, PMF)
- Transform (Laplace, z, moment generating function $E(e^{X\theta})$)
- Series of moments (if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{E(X^n)}} = \infty$)

Syllabus: Moments and transforms

Relation of moments and transforms:

- moment generating function:

$$E(e^{X\theta}) = E\left(\sum_{n=0}^{\infty} \frac{(X\theta)^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{E(X^n)\theta^n}{n!}$$

$$\longrightarrow \quad E(X^n) = \frac{d^n}{d\theta^n} E(e^{X\theta})\bigg|_{\theta=0}$$

- Laplace transform:

$$\frac{d^n}{ds^n} f^*(s)\bigg|_{s=0} = \frac{d^n}{ds^n} \int_t e^{-st} f(t)dt\bigg|_{s=0} =$$

$$\int_t (-t)^n e^{-st} f(t)dt\bigg|_{s=0} = (-1)^n \int_t t^n f(t)dt$$

$$\longrightarrow \quad E(X^n) = (-1)^n \frac{d^n}{ds^n} f^*(s)\bigg|_{s=0}$$

Syllabus: Moments and transforms

Relation of moments and transforms:

- z transform

$$\frac{d^n}{dz^n} N(z)\bigg|_{z=1} = \frac{d^n}{dz^n} \sum_{i=0}^{\infty} p_i \; z^i \bigg|_{z=1} =$$

$$\sum_{i=0}^{\infty} p_i \; i(i-1)\ldots(i-n+1) z^{i-n} \bigg|_{z=1} = \sum_{i=n}^{\infty} p_i \; \frac{i!}{(i-n)!}$$

Factorial moments:

$$\longrightarrow \quad E(X(X-1)\ldots(X-n+1)) = \frac{d^n}{dz^n} N(z)\bigg|_{z=1}$$

Syllabus: Conditional probability

Conditional probability: $Pr(A|B) = \dfrac{Pr(AB)}{Pr(B)}$,

Unconditioning (total probability):

$Pr(A) = \sum_i Pr(A|B_i)Pr(B_i)$ where $\sum_i Pr(B_i) = 1$

$Pr(A) = \int_x Pr(A|x)dF(x)$ where $\int_x dF(x) = 1$

Syllabus: Continuous distributions

Exponential distribution:

$f(t) = \lambda e^{-\lambda t}$, $F(t) = 1 - e^{-\lambda t}$, $\lambda(t) = \lambda$,

$$E(X) = \frac{1}{\lambda},\ c^2 = \frac{\sigma^2(X)}{E^2(X)} = \frac{E(X^2) - E^2(X)}{E^2(X)} = 1.$$

$$F^{\sim}(s) = E(e^{-sX}) = \int_t e^{-st} dF(t) = \frac{\lambda}{s + \lambda}$$

Erlang(n) distribution:

$$f(t) = \frac{\lambda(\lambda t)^{n-1}}{(n-1)!}\ e^{-\lambda t},\ E(X) = \frac{n}{\lambda},\ F^{\sim}(s) = \left(\frac{\lambda}{s+\lambda}\right)^n.$$

Syllabus: Discrete distributions

Geometric distribution $N \in \{1, 2, 3, \ldots\}$:

$$p_i = Pr(N = i) = (1 - p)p^{i-1}, \; E(N) = \frac{1}{p},$$

Geometric distribution $N' \in \{0, 1, 2, \ldots\}$:

$$p_i = Pr(N' = i) = (1 - p)p^i, \; E(N) = \frac{1}{p} - 1,$$

Poisson distribution $N \in \{0, 1, 2, \ldots\}$:

$$p_i = Pr(N = i) = \frac{\lambda^i}{i!}e^{-\lambda}, \; E(N) = \lambda,$$

Binomial distribution $N \in \{0, 1, 2, \ldots, n\}$:

$$p_i = Pr(N = i) = \binom{n}{i}p^i(1 - p)^{n-i}, \; E(N) = np,$$

Syllabus: Poisson process

3 identical representations:

- short term behaviour:

  $Pr(0 \text{ arrival in } (t, t+\delta)) = 1 - \lambda\delta + \sigma(\delta)$

  $Pr(1 \text{ arrival in } (t, t+\delta)) = \lambda\delta + \sigma(\delta)$

  $Pr(\text{more than 1 arrivals in}(t, t+\delta)) = \sigma(\delta)$

- inter-arrival time:

  inter-arrival periods are independent and exponentially distributed with parameter λ

  $\rightarrow$ time to the nth is Erlang(n) distributed.

- arrivals in t long interval:

  number of arrivals in any t long interval is Poisson distributed with parameter λt

  $$Pr(k \text{ arrivals in } (u, u+t)) = \frac{(\lambda t)^k}{k!}\, e^{-\lambda t}$$

Syllabus: Basic rules

Sum of discrete random variables ($Z = X + Y$):

$$z_i = \sum_k x_k y_{i-k}, \qquad Z(z) = X(z)Y(z)$$

Sum of continuous random variables ($Z = X + Y$):

$$f_Z(t) = \int_x f_X(x) f_Y(t-x)dx, \qquad F_Z^{\sim}(s) = F_X^{\sim}(s) F_Y^{\sim}(s)$$

Sum of random variables ($Z = X + Y$):

$$F_Z(t) = \int_x F_X(t-x)dF_Y(x), \qquad F_Z^{\sim}(s) = F_X^{\sim}(s) F_Y^{\sim}(s)$$

Remaining lifetime:

$$F_\tau(t) = Pr(X - \tau < t | X > \tau) = \frac{F(t+\tau) - F(\tau)}{1 - F(\tau)}$$

Equilibrium distribution of X:

$$f(t) = \frac{1 - F_X(t)}{E(X)}, \qquad E(Y^n) = \frac{E(X^{n+1})}{(n+1)E(X)}$$

Syllabus: Properties of distribution

Ageless distribution:

$\lambda(t)$ is constant
$\rightarrow$ exponential distribution, $c^2 = 1$

Aging distributions:

$\lambda(t)$ is increasing
$\rightarrow$ e.g., Erlang(n) distribution, $c^2 < 1$

Deaging distributions:

$\lambda(t)$ is decreasing
$\rightarrow$ e.g., Hyper-exponential distribution, $c^2 > 1$

Syllabus: Semi-Markov process

Time homogeneous discrete state continuous time stochastic process ($X(t)$) which is memoryless at state transition epochs ($T_0 = 0, T_1, T_2, \ldots$).

Kernel $K_{ij}(t) = Pr(T_1 < t, X(T_1) = j | X(0) = i)$ describes the joint distribution of the next state and the time spent in the current state.

The state of the process at state transitions form an "embedded" DTMC $X(T_0), X(T_1), X(T_2), X(T_3), \ldots$.

The state transition probability matrix of the embedded DTMC is $P = K(\infty)$. Let the stationary distribution of the embedded DTMC be π, that is $\pi P = \pi, \pi \mathbb{1} = 1$.

The distribution of time spent in state i is $K_i(t) \doteq Pr(T_1 < t | X(0) = i) = \sum_j K_{ij}(t)$ and its mean is $\tau_i = E(T_1 | X(0) = i) = \int_t 1 - K_i(t) dt$.

Transient distribution

$$z_i = \lim_{T \to \infty} Pr(X(T) = i) = \frac{\pi_i \tau_i}{\sum_j \pi_j \tau_j}$$

Syllabus: Semi-Markov process

Based on the ergodicity of semi-Markov processes we can write

$$
\begin{aligned}
z_i &= \lim_{T\to\infty} Pr(X(T)=i) = \lim_{T\to\infty} \frac{1}{T}\int_{t=0}^{T} \mathcal{I}_{\{X(t)=i\}} dt \\
&= \lim_{N\to\infty} \frac{1}{T_N}\int_{t=0}^{T_N} \mathcal{I}_{\{X(t)=i\}} dt \\
&= \lim_{N\to\infty} \frac{1}{T_N}\sum_{k=1}^{N}\int_{t=T_{k-1}}^{T_k} \mathcal{I}_{\{X(t)=i\}} dt \\
&= \lim_{N\to\infty} \frac{1}{T_N}\sum_{k=1}^{N} \mathcal{I}_{\{X(T_{k-1})=i\}}(T_k - T_{k-1}|X(T_{k-1})=i) \\
&= \lim_{N\to\infty} \frac{\underbrace{\sum_{k=1}^{N} \mathcal{I}_{\{X(T_{k-1})=i\}}(T_k - T_{k-1}|X(T_{k-1})=i)}_{\to N\pi_i\tau_i}}{\sum_j \underbrace{\sum_{k=1}^{N} \mathcal{I}_{\{X(T_{k-1})=j\}}(T_k - T_{k-1}|X(T_{k-1})=j)}_{\to N\pi_j\tau_j}} \\
&= \frac{\pi_i\tau_i}{\sum_j \pi_j\tau_j}
\end{aligned}
$$

Syllabus: Markov regenerative process

Time homogeneous discrete state continuous time stochastic process ($X(t)$) which is memoryless at some instance of time ($T_0 = 0, T_1, T_2, \ldots$).

The global kernel, $K_{ij}(t) = Pr(T_1 < t, X(T_1) = j | X(0) = i)$, describes the joint distribution of the state at the next memoryless instance and the time to the next memoryless instance.

The process behaviour between memoryless instances is described the local kernel $E_{ij}(t) = Pr(T_1 > t, X(t) = j | X(0) = i)$.

The state of the process at memoryless instances form an "embedded" DTMC $X(T_0), X(T_1), X(T_2), X(T_3), \ldots$.

The state transition probability matrix of the embedded DTMC is $P = K(\infty)$. Let the stationary distribution of the embedded DTMC be π, that is $\pi P = \pi, \pi \mathbb{1} = 1$.

During a regenerative period starting from i the mean time spent in state j is $\tau_{ij} = \int_t E_{ij}(t)dt$.

Transient distribution

$$z_i = \lim_{T \to \infty} Pr(X(T) = i) = \frac{\sum_j \pi_j \tau_{j,i}}{\sum_j \sum_k \pi_j \tau_{j,k}}$$

Syllabus: Markov regenerative process

Based on the ergodicity of Markov regenerative processes we can write

$$z_i = \lim_{T\to\infty} Pr(X(T)=i) = \lim_{T\to\infty} \frac{1}{T}\int_{t=0}^{T} \mathcal{I}_{\{X(t)=i\}}dt$$

$$= \lim_{N\to\infty} \frac{1}{T_N}\int_{t=0}^{T_N} \mathcal{I}_{\{X(t)=i\}}dt$$

$$= \lim_{N\to\infty} \frac{1}{T_N}\sum_{k=1}^{N}\int_{t=T_{k-1}}^{T_k} \mathcal{I}_{\{X(t)=i\}}dt$$

$$= \lim_{N\to\infty} \frac{1}{T_N}\sum_j\sum_{k=1}^{N}\mathcal{I}_{\{X(T_{k-1})=j\}}\int_{t=T_{k-1}}^{T_k} \mathcal{I}_{\{X(t)=i|X(T_{k-1})=j\}}dt$$

$$= \lim_{N\to\infty} \frac{\sum_j \underbrace{\sum_{k=1}^{N}\mathcal{I}_{\{X(T_{k-1})=j\}}\int_{t=T_{k-1}}^{T_k} \mathcal{I}_{\{X(t)=i|X(T_{k-1})=j\}}dt}_{\to N\pi_j\tau_{ji}}}{\sum_j\sum_k \underbrace{\sum_{k=1}^{N}\mathcal{I}_{\{X(T_{k-1})=j\}}\int_{t=T_{k-1}}^{T_k} \mathcal{I}_{\{X(t)=k|X(T_{k-1})=j\}}dt}_{\to N\pi_j\tau_{jk}}}$$

$$= \frac{\sum_j \pi_j\tau_{ji}}{\sum_j\sum_k \pi_j\tau_{j,k}}$$

M/G/1 queue

Poisson arrival process, general service time distribution, one server, infinite buffer, FIFO.

$\rightarrow X(t)$ is not a CTMC.

System behaviour depends on elapsed service time of customer under service.

Memoryless instances: e.g. departure instances.

$\rightarrow$ embedded Discrete time Markov chain

Notations:

λ arrival rate, B service time r.v. ($\overline{T_B} = E(B)$),

Q queue length r.v., W waiting time r.v.,

W_0 remaining service time r.v.

M/G/1 queue: mean queue length

Server utilization: $\rho = \lambda \overline{T_B}$

Mean waiting time:

$$\overline{W} = \overline{W_0} + \overline{Q}\ \overline{T_B}$$

Little's law ($\overline{Q} = \lambda \overline{W}$) $\rightarrow$

$$\overline{W} = \frac{\overline{W_0}}{1-\rho}$$

Remaining service time of customer under service:

$$\overline{W_0} = P(server\ busy)\ \overline{R} + P(server\ idle)\ 0 = \rho\ \overline{R}$$

Remaining service time of busy server:

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T_B}} = \frac{\overline{T_B}}{2}(1 + c_B^2)$$

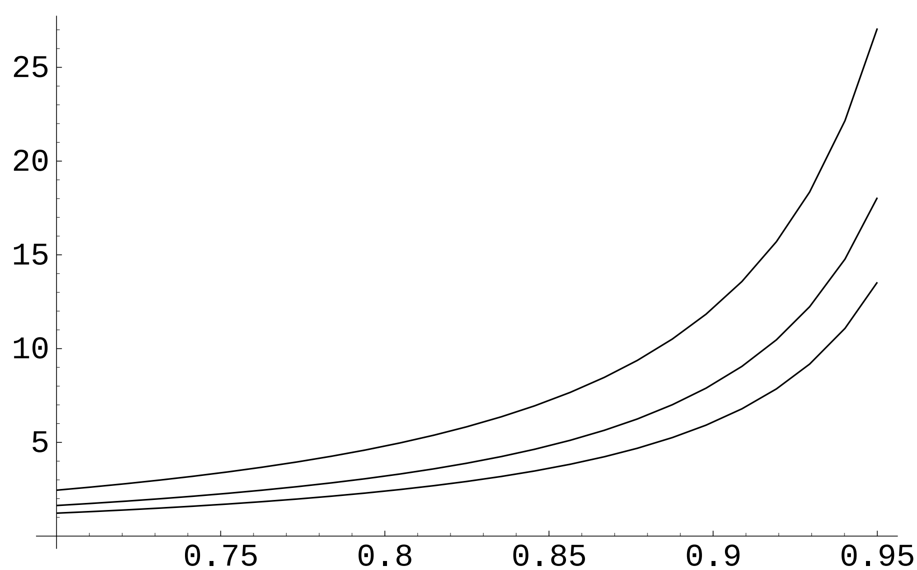
Applying Little's law again $\rightarrow$

$$\overline{Q} = \lambda \overline{W} = \frac{\rho^2(1 + c_B^2)}{2(1-\rho)}$$

Pollaczek-Khinchin formulae for mean queue length.

M/G/1 queue: mean queue length

Mean queue length ($\overline{Q}$) versus utilization (ρ) with $c_B^2 = 0.5, 1, 2$



M/G/1 queue: stationary distribution

DTMC embedded in departure epochs:

X_n number of customers after the nth departure

$$X_{n+1} = \begin{cases} X_n - 1 + Y, & \text{if } X_n > 0 \\ Y, & \text{if } X_n = 0 \end{cases}$$

$$X_{n+1} = (X_n - 1)^+ + Y$$

Transition probability matrix:

$$P = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ 0 & a_0 & a_1 & \cdots \\ 0 & 0 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The number of customer arrives during a service period:

$$a_k = P(k \text{ customer arrives}) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} dB(t)$$

M/G/1 queue: stationary distribution

Balance equations of the embedded DTMC:

$$\nu_0 = \nu_0 a_0 + \nu_1 a_0$$

$$\nu_k = \nu_0 a_k + \sum_{i=1}^{k+1} \nu_i a_{k-i+1}, \qquad k \geq 1$$

Multiplying the kth equation by z^k and summing up results:

$$G(z) = \sum_{k=0}^{\infty} \nu_k z^k = \nu_0 G_A(z) + \frac{1}{z}(G(z) - \nu_0) G_A(z),$$

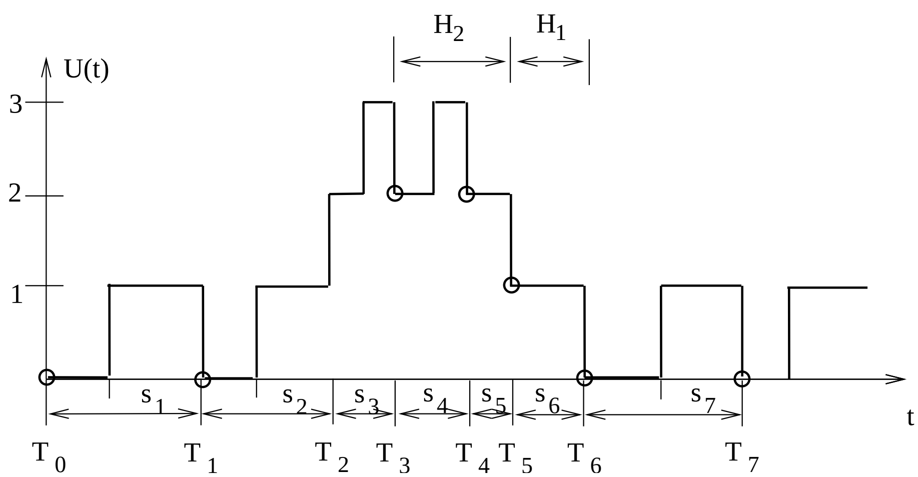
where

$$\begin{aligned} G_A(z) &= \sum_{k=0}^{\infty} a_k z^k = \ldots \\ &= \int_{t=0}^{\infty} e^{-\lambda t(1-z)} dB(t) = B^{\sim}(\lambda(1-z)) \end{aligned}$$

Pollaczek-Khinchin formulae for queue length distribution:

$$G(z) = \nu_0 \frac{(z-1) B^{\sim}(\lambda(1-z))}{z - B^{\sim}(\lambda(1-z))}$$

M/G/1 queue: busy period



The number of customer served between a departure with n customers and the first time with $n-1$ customers is Q_n.

First time to have $n-1$ customers in the system starting from a departure with n customers is H_n.

Due to the regular "level independent" structure of the M/G/1 queue and matrix P H_n and Q_n are independent of n !!!

M/G/1 queue: busy period

The number of customers arrive during the service time suppose that $B = \tau$ is Poisson distributed with $\lambda\tau$.

The conditional distribution of the busy period is

$$H|B = \tau \;\; = \begin{cases} \tau & e^{\lambda\tau} \\ \tau + H_1 & \lambda\tau e^{\lambda\tau} \\ \tau + H_2 + H_1 & \frac{(\lambda\tau)^2}{2!} e^{\lambda\tau} \\ \ldots & \end{cases}$$

Using $h^*(s) = E(e^{-sH}) = h_1^*(s) = h_2^*(s) = \ldots$ we have

$$\begin{aligned} E(e^{-sH}|B = \tau) \;\; &= \sum_{i=0}^{\infty} \frac{(\lambda\tau)^i}{i!} e^{-\lambda\tau} e^{-s\tau} (h^*(s))^i \\ &= e^{-\lambda\tau} e^{-s\tau} e^{\lambda\tau h^*(s)} = e^{-\tau(s+\lambda-\lambda h^*(s))} \end{aligned}$$

and

$$\begin{aligned} h^*(s) \;\; &= \int_{\tau=0}^{\infty} E(e^{-sH}|B = \tau) b(\tau) d\tau = \\ &= \int_{\tau=0}^{\infty} e^{-\tau(s+\lambda-\lambda h^*(s))} b(\tau) d\tau = \\ &= b^*(s + \lambda - \lambda h^*(s)) \end{aligned}$$

M/G/1 queue: busy period

Similarly the conditional distribution of the number of customers served in busy period is

$$Q|B = \tau \; = \begin{cases} 1 & e^{\lambda\tau} \\ 1 + Q_1 & \lambda\tau e^{\lambda\tau} \\ 1 + Q_2 + Q_1 & \frac{(\lambda\tau)^2}{2!} e^{\lambda\tau} \\ \ldots \end{cases}$$

Using $Q(z) = E(z^Q) = Q_1(z) = Q_2(z) = \ldots$ it is

$$\begin{aligned} E(z^Q|B = \tau) &= z \sum_{i=0}^{\infty} \frac{(\lambda\tau)^i}{i!} e^{-\lambda\tau} Q(z)^i \\ &= z \; e^{-\lambda\tau} e^{\lambda\tau Q(z)} = z \; e^{-\tau(\lambda - \lambda Q(z))} \end{aligned}$$

and

$$\begin{aligned} Q(z) &= \int_{\tau=0}^{\infty} E(z^Q|B = \tau) b(\tau) d\tau = \\ &= \int_{\tau=0}^{\infty} z \; e^{-\tau(\lambda - \lambda Q(z))} b(\tau) d\tau = \\ &= z \; b^*(\lambda(1 - Q(z))) \end{aligned}$$

M/G/1 queue: busy period

The moments of H and Q can be obtained from $h^*(s)$ and $Q(z)$. E.g.,

$$\begin{aligned} E(H) &= -\frac{d}{ds}h^*(s)|_{s=0} \\ &= -b^{*\prime}(s+\lambda-\lambda h^*(s))(1-\lambda h^{*\prime}(s))|_{s=0} \\ &= -b^{*\prime}(0)(1-\lambda h^{*\prime}(0)) = \overline{T_B}(1+\lambda E(H)) \\ &= \frac{\overline{T_B}}{1-\rho} \ . \end{aligned}$$

Since $\rho = \lambda \overline{T_B}$.

But they can be calculated directly as well:

$$E(H|B=\tau) = \tau + \sum_{i=0}^{\infty} \frac{(\lambda\tau)^i}{i!} e^{-\lambda\tau} i E(H) = \tau + \lambda\tau E(H)$$

and

$$\begin{aligned} E(H) &= \int_{\tau=0}^{\infty} E(H|B=\tau) b(\tau) d\tau \\ &= (1+\lambda E(H)) \int_{\tau=0}^{\infty} \tau b(\tau) d\tau \\ &= (1+\lambda E(H)) \overline{T_B} \\ &= \frac{\overline{T_B}}{1-\rho} \ . \end{aligned}$$

Similarly

$$E(Q) = \frac{1}{1-\rho} \ .$$

M/G/1 queue: special cases

M/M/1 queue: $B^{\sim}(s) = \dfrac{\mu}{s + \mu}$, $c_B^2 = 1$

$$\overline{Q} = \frac{\rho^2}{1 - \rho}$$

$$G(z) = \nu_0 \; \frac{1}{1 - \rho z} = \frac{1 - \rho}{1 - \rho z}$$

$$\nu_k = \nu_0 \rho^k = (1 - \rho)\rho^k$$

M/D/1 queue: $B^{\sim}(s) = e^{-sD}$, $c_B^2 = 0$, $(\rho = \lambda D)$

$$\overline{Q} = \frac{\rho^2}{2(1 - \rho)}$$

$$G(z) = \nu_0 \; \frac{z - 1}{z e^{\rho(1-z)} - 1}$$

M/G/1 queue as Markov regenerative process

$X(t)$ is the number of customers at time t. There are embedded time points, $T_0, T_1, \ldots$, at customer departures. The global and local kernels are

$$\begin{aligned} K_{ij}(t) &= Pr(T_1 < t, X(T_1) = j | X(0) = i), \\ E_{ij}(t) &= Pr(T_1 > t, X(t) = j | X(0) = i). \end{aligned}$$

$i = 1$:

$$\begin{aligned} K_{1k}(t) &= Pr(B < t, k \text{ arrivals in } (0, B)) \\ &= \int_{\tau=0}^{t} \frac{(\lambda\tau)^k}{k!} \, e^{-\lambda\tau} dB(\tau), \quad k \geq 0, \end{aligned}$$

$$\begin{aligned} E_{1k}(t) &= Pr(B > t, k-1 \text{ arrivals in } (0, t)) \\ &= \frac{(\lambda t)^{k-1}}{(k-1)!} \, e^{-\lambda t} \, (1 - B(t)), \quad k \geq 1, \end{aligned}$$

M/G/1 queue as Markov regenerative process

$i = 0$:

$$E_{00}(t) \;\; = Pr(0 \text{ arrival in } (0,t)) = e^{-\lambda t}$$

first arrival at τ and a service period starting from 1:

$$E_{0j}(t) \;\; = \int_{\tau=0}^{t} \lambda e^{-\lambda\tau} E_{1j}(t-\tau) d\tau, \quad j \geq 1,$$

$$K_{0j}(t) \;\; = \int_{\tau=0}^{t} \lambda e^{-\lambda\tau} K_{1j}(t-\tau) d\tau, \quad j \geq 0,$$

$i > 1$:

$$K_{ij}(t) = \begin{cases} K_{1,j-i+1}(t), & \text{if } i \geq 1, j \geq i-1, \\ 0, & \text{othervise.} \end{cases}$$

$$E_{ij}(t) = \begin{cases} E_{1,j-i+1}(t), & \text{if } i \geq 1, j \geq i, \\ 0, & \text{othervise.} \end{cases}$$

Exercise: relation of the embedded and the stationary distribution based on this MRP representation.

G/M/1 queue

Renewal arrival process (i.i.d. inter-arrival times), exponentially distributed service time, m server, infinite buffer, FIFO.

$\rightarrow X(t)$ is not a CTMC.

System behaviour depends on the time elapsed since the last arrival.

Memoryless instances: arrival instances.

$\rightarrow$ embedded Discrete time Markov chain

Special case: G/M/1 queue

DTMC embedded in arrival epochs:

X_n number of customers before the nth arrival

$$X_{n+1} = X_n + 1 - Y' = (X_n + 1 - Y)^+$$

where

- Y' number of customer served between the nth and $n+1$th arrivals,
- Y number of Poisson(μ) instances between the nth and $n+1$th arrivals.

Special case: G/M/1 queue

Transition probability matrix:

$$P = \begin{pmatrix} c_0 & b_0 & 0 & 0 & \cdots \\ c_1 & b_1 & b_0 & 0 & \cdots \\ c_2 & b_2 & b_1 & b_0 & \cdots \\ c_3 & b_3 & b_2 & b_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The number of Poisson(μ) instances during an inter-arrival period:

$$b_k = \int_0^\infty e^{-\mu t} \frac{(\mu t)^k}{k!} dA(t)$$

The more than k Poisson(μ) instances during an inter-arrival period:

$$c_k = \sum_{i=k+1}^{\infty} b_i$$

G/M/m queue

Server utilization is $\rho = \dfrac{\overline{\lambda}}{m\mu}$,
where $\overline{\lambda}$ is the mean arrival rate.

Transition probability matrix:

$$P = \left(\begin{array}{ccccc|cc}
p_{00} & p_{01} & 0 & \cdots & 0 & 0 & \cdots \\
p_{10} & p_{11} & p_{12} & \cdots & 0 & 0 & \cdots \\
p_{20} & p_{21} & p_{22} & \cdots & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & & 0 & 0 & \cdots \\
p_{m-1,0} & p_{m-1,1} & p_{m-1,2} & \cdots & b_0 & 0 & \cdots \\
p_{m,0} & p_{m,1} & p_{m,2} & \cdots & b_1 & b_0 & \cdots \\
\hline
p_{m+1,0} & p_{m+1,1} & p_{m+1,2} & \cdots & b_2 & b_1 & \cdots \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \ddots
\end{array} \right)$$

Number of services when all servers are busy:

$$b_k = \int_0^\infty e^{-m\mu t} \frac{(m\mu t)^k}{k!} \; dA(t)$$

G/M/m queue

After an arrival $i+1 \leq m$ customers:
all customers are under service
$i-j+1$ complete service, j do not:

$$p_{i,j} = \int_0^\infty \binom{i+1}{i-j+1}(1-e^{-\mu t})^{i-j+1} e^{-\mu t j} dA(t)$$

After the arrival $i+1 > m$ customers:
$i-m+1$ customers are in queue, m under service.
τ : time to empty the queue
(Erlang(i-m+1) distribution)

$$p_{i,j} = \int_{t=0}^\infty \int_{x=0}^t \binom{m}{j}(1-e^{-\mu(t-x)})^{m-j} e^{-\mu(t-x)j} f_\tau(x) dx dA(t)$$

where τ is Erlang$(i-m+1, m\mu)$, that is

$$f_\tau(x) = \frac{m\mu(m\mu x)^{i-m}}{(i-m)!}\ e^{-m\mu x}.$$

G/M/m queue: stationary distribution

Conjecture: geometric stationary distribution

$$\nu_0, \nu_1, \ldots, \nu_{m-2}, \kappa\sigma^{m-1}, \kappa\sigma^m, \kappa\sigma^{m+1}, \ldots$$

Verification ($k \geq m$):

$$\nu_k = \nu_{k-1} b_0 + \nu_k b_1 + \ldots = \sum_{i=k-1}^{\infty} \nu_i b_{i-k+1}$$

Using $\nu_k = \kappa\sigma^k$: $\quad \kappa\sigma^k = \sum_{i=k-1}^{\infty} \kappa\sigma^i b_{i-k+1}$

Hence

$$\sigma = \sum_{i=0}^{\infty} \sigma^i b_i = \int_0^\infty e^{-m\mu t} \sum_{i=0}^{\infty} \frac{(\sigma m\mu t)^i}{i!}\, dA(t) =$$
$$\int_0^\infty e^{-(m\mu - m\mu\sigma)t}\, dA(t) = A^{\sim}(m\mu - m\mu\sigma),$$

that is

$$\sigma = A^{\sim}(m\mu - m\mu\sigma).$$

The $\nu_0, \nu_1, \ldots, \nu_{m-2}$ state probabilities and κ are obtained from the linear system of the first m equations.

G/M/m queue: waiting time

Probability of queueing an arriving customer:

$$Pr(queueing) = \sum_{i=m}^{\infty} \nu_i = \sum_{i=m}^{\infty} \kappa\sigma^i = \frac{\kappa\sigma^m}{1-\sigma}$$

Queue length distribution (prior to arrival) if arriving customer joints queue:

$$Pr(Q = k|queueing) = \frac{\kappa\sigma^{m+k}}{\frac{\kappa\sigma^m}{1-\sigma}} = (1-\sigma)\sigma^k$$

Waiting time distribution if $n-m$ customers enqueue prior to arrival:

$$W^{\sim}(s|n-m) = \left(\frac{m\mu}{s+m\mu}\right)^{n-m+1}$$

Waiting time distribution if the customers queues:

$$W^{\sim}(s|queueing) = \sum_{n=m}^{\infty} W^{\sim}(s|n-m)Pr(Q = n-m|queueing) = \frac{(1-\sigma)m\mu}{s+(1-\sigma)m\mu}$$

Exponentially distributed with parameter $(1-\sigma)m\mu$.

G/M/1 queue: ν_k versus π_k

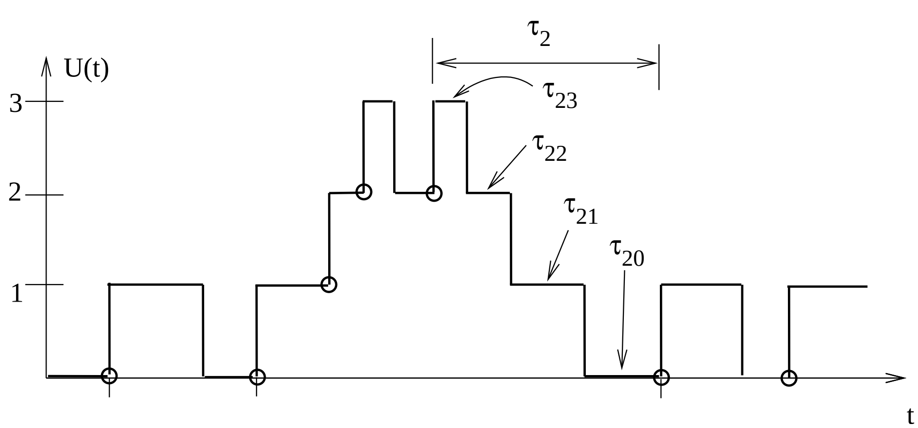
$X(t)$ is a Markov regenerative process.

Exercise: global and local kernels of the MRP embedded at arrival instances.

The stationary distribution, can be computed as:

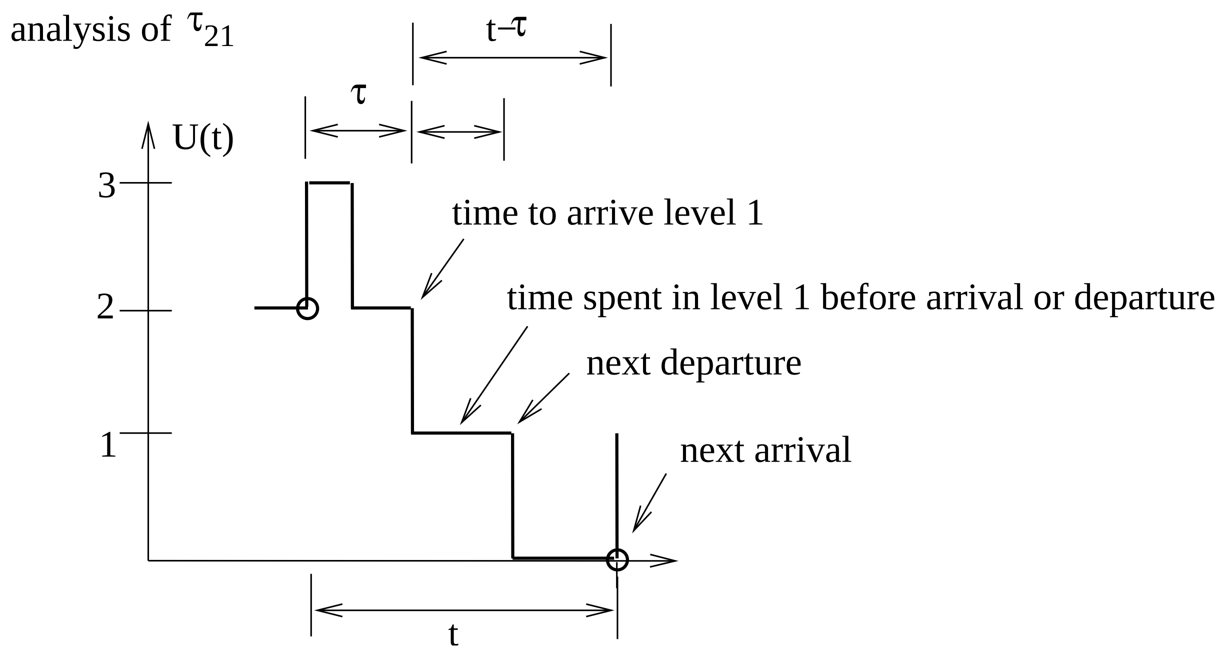
$$\pi_k = \frac{\sum_j \nu_j \tau_{jk}}{\sum_j \nu_j \tau_j}$$

where τ_j is the mean time to the next embedded instance starting from state j, and τ_{jk} is the mean time spent in state k before the next embedded instance starting from state j.



$\tau_i = 1/\bar{\lambda}$, since the time to the next embedded instance is an interarrival time.

G/M/1 queue: ν_k versus π_k



Analysis of τ_{ik}:

τ is the sum of $i+1-k$ service times

$\rightarrow$ Erlang$(i+1-k, \nu)$ distribution

$$\tau_{ik} = \int_{t=0}^{\infty} \int_{\tau=0}^{t} \int_{x=0}^{t-\tau} e^{-\mu x}\ dx\ f_{Erl(i+1-k)}(\tau)\ d\tau\ dA(t)$$

$$\tau_{i0} = \int_{t=0}^{\infty} \int_{\tau=0}^{t} (t-\tau)\ f_{Erl(i+1-k)}(\tau)\ d\tau\ dA(t)$$

Level independent behaviour ($k>0$): $\tau_{i,k} = \tau_{i+j,k+j}, \quad \forall j$

G/M/1 queue

Single server: $m = 1$
$\rightarrow \nu_k = \kappa\sigma^k = (1-\sigma)\sigma^k$

PASTA property does not hold:

$$\nu_0 = 1 - \sigma \neq \pi_0 = 1 - \rho = 1 - \frac{\overline{\lambda}}{m\mu}$$

Indeed $\pi_0 = 1 - \rho$ and $\pi_k = \rho(1-\sigma)\sigma^{k-1}$ $(k \geq 1)$.

Queue parameters:

$$\overline{K} = \frac{\rho}{1-\sigma} \qquad \overline{T} = \frac{1}{\mu}\,\frac{1}{1-\sigma}$$

$$\overline{Q} = \frac{\rho\ \sigma}{1-\sigma} \qquad \overline{W} = \frac{1}{\mu}\,\frac{\sigma}{1-\sigma}$$

Special G/M/1 queues

M/M/1 queue: $A^{\sim}(s) = \dfrac{\lambda}{s+\lambda}$

$$\sigma = A^{\sim}(\mu - \sigma\mu) = \frac{\lambda}{\mu - \sigma\mu + \lambda}$$

$$\sigma_1 = \frac{\lambda}{\mu} = \rho \qquad (\sigma_2 = 1)$$

E$_2$/M/1 queue: $A^{\sim}(s) = \left(\dfrac{\lambda}{s+\lambda}\right)^2$

$$\rho = \frac{\overline{\lambda}}{\mu} = \frac{\lambda}{2\mu}, \qquad \sigma = 2\rho + \frac{1}{2} - \sqrt{2\rho + \frac{1}{4}}$$

D/M/1 queue: $A^{\sim}(s) = e^{-sD}$

$$\rho = \mu D, \qquad \sigma = A^{\sim}(\mu - \sigma\mu) = e^{-\mu D(1-\sigma)}$$

H$_2$/M/1 queue: $A^{\sim}(s) = \dfrac{p_1\lambda_1}{s+\lambda_1} + \dfrac{p_2\lambda_2}{s+\lambda_2}$

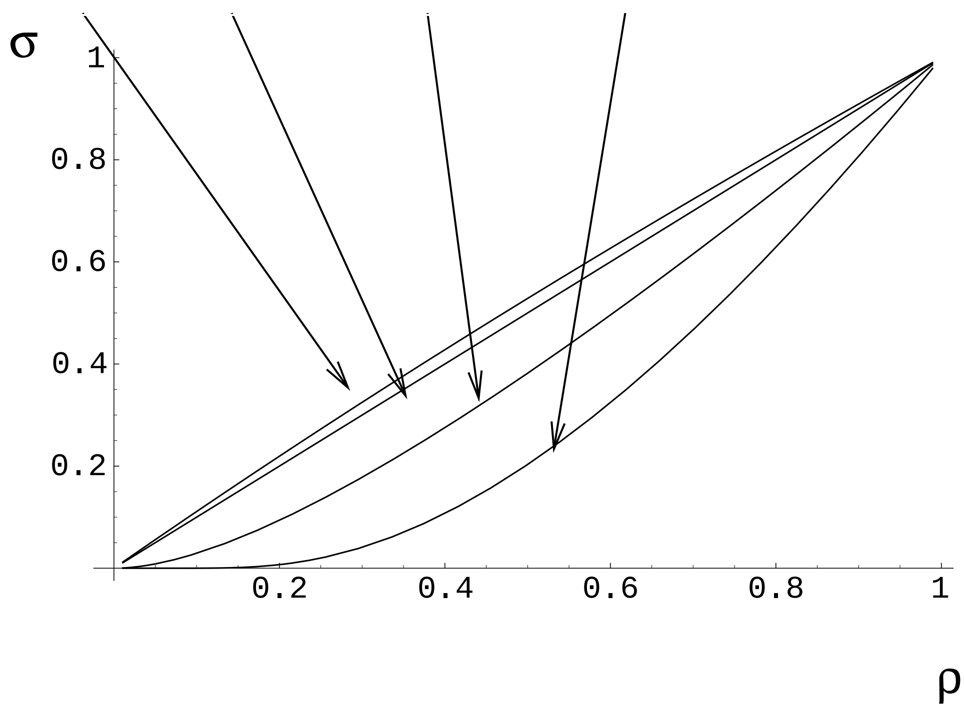
$$p_1 = p_2 = 0.5, \quad \lambda_1 = 2\lambda_2 = \lambda = 1, \quad \overline{\lambda} = \frac{2\lambda}{3}$$

$$\rho = \frac{\overline{\lambda}}{\mu} = \frac{2\lambda}{3\mu} \qquad \sigma = \frac{9\rho}{8} + \frac{1}{2} - \sqrt{\frac{9\rho^2}{64} + \frac{1}{4}}$$

Special G/M/1 queues

σ versus ρ

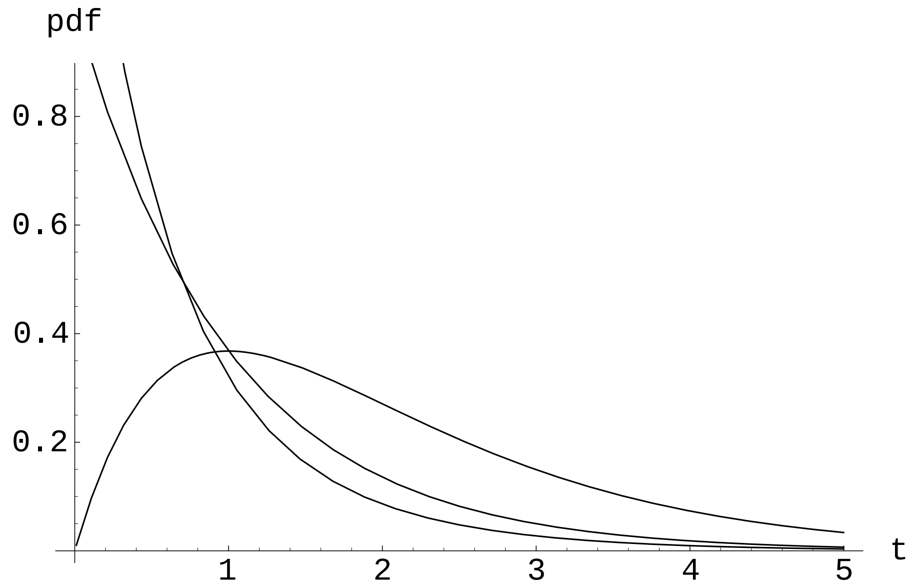
in H_2/M/1, M/M/1, E_2/M/1, D/M/1 queues

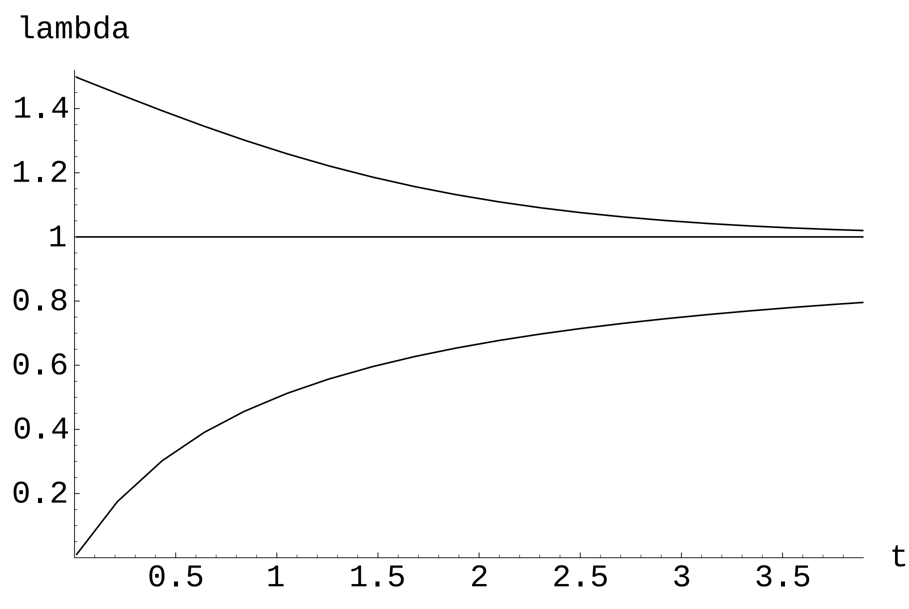


Special G/M/1 queues

Relation of inter-arrival time distributions:

$c^2_{H_2} = 1.22 > c^2_{Exp} = 1 > c^2_{E_2} = 0.5 > c^2_{Det} = 0$





G/G/1 queue

Renewal arrival process, general service time distribution, one server, infinite buffer, FIFO.

$\rightarrow X(t)$ is not a CTMC.

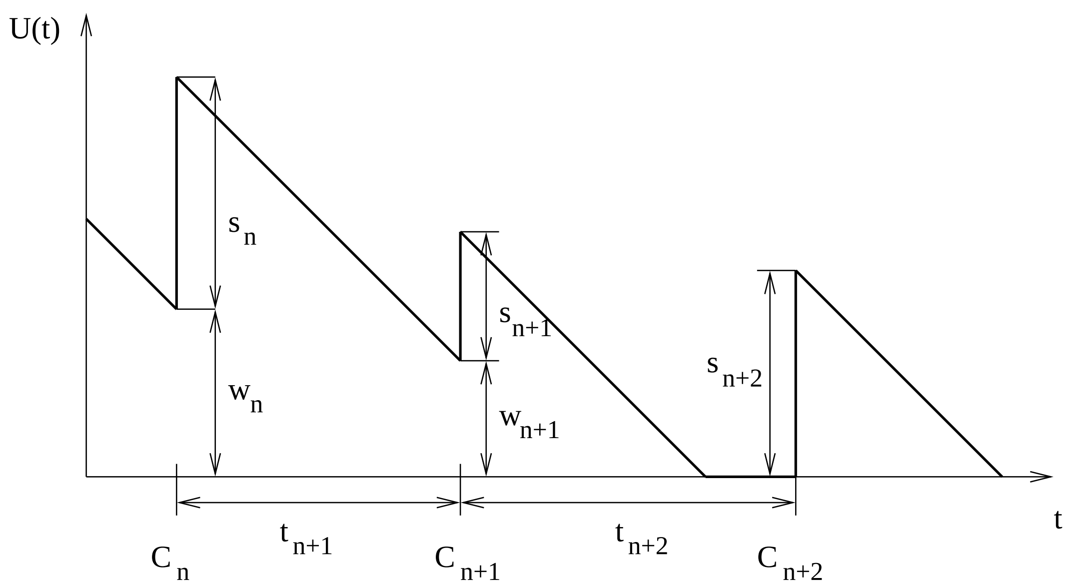
System behaviour depends on service time of customer under service and the last arrival time.

G/G/1 queue: Unfinished work

s_n service time of the nth customer,

t_{n+1} inter-arrival time after the nth customer.

Unfinished work, $U(t)$: the amount of time to complete the service of customers in the system.



$$w_{n+1} = \begin{cases} w_n + s_n - t_{n+1} & \text{if } w_n + s_n - t_{n+1} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $u_n = s_n - t_{n+1}$

$$w_{n+1} = max(0, w_n + u_n) = (w_n + u_n)^+$$

Stability condition: $E(u_n) < 0$

G/G/1 queue: Lindley integral equation

Distribution of u_n:

$$C_n(x) = Pr(u_n \leq x) = \int_{t=0}^{\infty} B(t+x)\ dA(t)$$

Distribution of w_{n+1}:

$$W_{n+1}(x) = Pr(w_{n+1} \leq x) = \int_{t=0^-}^{\infty} C_n(x-t)\ dW_n(t)$$

$(W_n(x) = 0$ for $x < 0.)$

Stationary behaviour $(n \to \infty)$

$$W(x) = Pr(w \leq x) = \int_{t=0^-}^{\infty} C(x-t)\ dW(t)$$

Lindley integral equation

G/G/1 queue: Lindley integral equation

Solution of Lindley integral equation:

Spectral solution, based on $A^{\sim}(s)$ and $B^{\sim}(s)$.

Numerical approximation:

$$
\begin{aligned}
w_1 &= max(0, u_0 + w_0) \\
w_2 &= max(0, u_1 + w_1) = max(0, u_1, u_1 + u_0 + w_0) \\
w_3 &= max(0, u_2 + w_2) \\
&= max(0, u_2, u_2 + u_1, u_2 + u_1 + u_0 + w_0),
\end{aligned}
$$

where u_n are i.i.d. random variables with $E(u_n) < 0$.

One can approximate $W(x)$ based on a finite series, since

$$\lim_{n\to\infty} Pr\left(\sum_{i=1}^{n} u_i > 0\right) = 0$$

Phase type distributions

Time to absorption in a Markov chain with N transient and 1 absorbing state.

If the Markov chain is

- CTMC $\rightarrow$ Continuous Phase Type distribution (CPH)
- DTMC $\rightarrow$ Discrete Phase Type distribution (DPH)

Representation:

Initial probability distribution ($\boldsymbol{\alpha}$) + Markov chain description

- CPH $\rightarrow$ generator matrix ($\boldsymbol{A}$)
- DPH $\rightarrow$ transition probability matrix ($\boldsymbol{B}$)

Only for transient states.

Properties of phase type distributions

CPH distributions:

Generator matrix: $\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{a} \\ \mathbf{0} & 0 \end{bmatrix}$ $\qquad (\mathbf{a} = -\mathbf{A}\mathbb{1})$

PDF: $f(t) = \boldsymbol{\alpha} e^{\mathbf{A}t} \mathbf{a}$

CDF: $F(t) = 1 - \boldsymbol{\alpha} e^{\mathbf{A}t} \mathbb{1}$

power moments: $\mu_k = k! \ \boldsymbol{\alpha}(-\mathbf{A})^{-k}\mathbb{1} = k! \ \boldsymbol{\alpha}(-\mathbf{A})^{-k-1}\mathbf{a}$

LST: $f^*(s) = \boldsymbol{\alpha}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{a} = \boldsymbol{\alpha} \left[\frac{det(s\mathbf{I} - \mathbf{A})_{ji}}{det(s\mathbf{I} - \mathbf{A})} \right] \mathbf{a}$

DPH distributions:

Generator matrix: $\hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix}$ $\qquad (\mathbf{b} = \mathbb{1} - \mathbf{B}\mathbb{1})$

PMF: $p_k = Pr(X = k) = \boldsymbol{\alpha}\mathbf{B}^{k-1}\mathbf{b}$

CDF: $F(k) = Pr(X \leq k) = 1 - \boldsymbol{\alpha}\mathbf{B}^k\mathbb{1}$

factorial moments: $\gamma_k = k! \ \boldsymbol{\alpha}(\mathbf{I} - \mathbf{B})^{-k}\mathbf{B}^{k-1}\mathbb{1}$

z-transform: $\mathcal{F}(z) = z \ \boldsymbol{\alpha}(\mathbf{I} - z\mathbf{B})^{-1}\mathbf{b} = z \ \boldsymbol{\alpha} \left[\frac{det(\mathbf{I} - z\mathbf{B})_{ji}}{det(\mathbf{I} - z\mathbf{B})} \right] \mathbf{b}$

Properties of phase type distributions

CPH	DPH
rational Laplace tr.	rational Z transform
closed for min/max, mixture, summation, ...	
$f(t) > 0$	$p_i = Pr(X = i) \geq 0$
infinite support	finite or infinite support
exponential tail	geometric tail
$CV_{min} = \dfrac{1}{N} > 0$	$CV_{min} = F(N, \mu) \geq 0$
$CV_{min} \leftrightarrow$ Erlang distr.	$CV_{min} \leftrightarrow$ Discrete Erlang or Determined structure

Operations with phase type distributions

Summation:

$Z = X + Y$, where X and Y are independent, X is PH($\alpha, \boldsymbol{A}$) and Y is PH($\beta, \boldsymbol{B}$)

then Z is PH($\gamma, \boldsymbol{G}$) with

$$\boldsymbol{\gamma} = \begin{bmatrix} \boldsymbol{\alpha} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{A} & \mathbf{a}\boldsymbol{\beta} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}$$

Operations with phase type distributions

Mixture:

$$Z = \begin{cases} X & \text{with probability } p, \\ Y & \text{with probability } (1-p), \end{cases}$$

where X and Y are independent, X is PH$(\alpha, \boldsymbol{A})$ and Y is PH$(\beta, \boldsymbol{B})$

then Z is PH$(\gamma, \boldsymbol{G})$ with

$$\boldsymbol{\gamma} = \left[\begin{array}{cc} p\boldsymbol{\alpha} & (1-p)\boldsymbol{\beta} \end{array} \right]$$

$$\mathbf{G} = \left[\begin{array}{cc} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{array} \right]$$

Operations with phase type distributions

Minimum:

$Z = Min(X, Y)$, where X and Y are independent, X is PH($\alpha, \boldsymbol{A}$) and Y is PH($\beta, \boldsymbol{B}$)

then Z is PH($\gamma, \boldsymbol{G}$) with

$$\boldsymbol{\gamma} = \boldsymbol{\alpha} \otimes \boldsymbol{\beta}$$

$$\mathbf{G} = \mathbf{A} \oplus \mathbf{B}$$

where

Kronecker product: $\mathbf{A} \bigotimes \mathbf{B} = \boxed{\begin{matrix} A_{11}\mathbf{B} & \dots & A_{1n}\mathbf{B} \\ \vdots & & \vdots \\ A_{n1}\mathbf{B} & \dots & A_{nn}\mathbf{B} \end{matrix}}$

Kronecker sum: $\mathbf{A} \bigoplus \mathbf{B} = \mathbf{A} \bigotimes \mathbf{I_B} + \mathbf{I_A} \bigotimes \mathbf{B}$

Operations with phase type distributions

Maximum:

$Z = Max(X, Y)$, X and Y are independent, where X is PH($\alpha, \boldsymbol{A}$) and Y is PH($\beta, \boldsymbol{B}$)

then Z is PH($\gamma, \boldsymbol{G}$) with

$$\boldsymbol{\gamma} = \left[\ \boldsymbol{\alpha} \otimes \boldsymbol{\beta} \mid \mathbf{0} \mid \mathbf{0}\ \right]$$

$$\mathbf{G} = \left[\begin{array}{c|c|c} \mathbf{A} \oplus \mathbf{B} & \mathbf{a} \oplus \mathbf{I} & \mathbf{I} \oplus \mathbf{b} \\ \hline \mathbf{0} & \mathbf{B} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{A} \end{array}\right]$$

Multi terminal phase type distributions

There is a Markov chain with N transient state and K absorbing ones, whose generator matrix is

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{a_1} & \dots & \mathbf{a_K} \\ \mathbf{0} & 0 & \dots & 0 \\ \mathbf{0} & \vdots & \ddots & \vdots \\ \mathbf{0} & 0 & \dots & 0 \end{bmatrix} \qquad \sum_{k=1}^{K} \mathbf{a_k} = -\mathbf{A}\mathbb{1}$$

T is the time to leave the transient group (first N states) and T_k is the time to reach absorbing state k. ($T = \min_k T_k$, if $T_k = T$ then $T_{j,j\neq k} = \infty$).

defective PDF of T_k:

$$f_{T_k}(t) = \lim_{\Delta\to 0} \frac{1}{\Delta} Pr(t \le T_k < t + \Delta) = \boldsymbol{\alpha} e^{\mathbf{A}t}\mathbf{a_k}$$

because

$$Pr(T_k = T) = \int_{t=0}^{\infty} f_k(t)dt = \boldsymbol{\alpha}(-\mathbf{A})^{-1}\mathbf{a_k}.$$

non-defective PDF of $T_k | T_k = \min_j T_j$:

$$f^c_{T_k}(t) = \frac{f_{T_k}(t)}{Pr(T_k = T)} =$$

$$\lim_{\Delta\to 0} \frac{1}{\Delta} Pr(t \le T_k < t + \Delta | T_k = \min_j T_j) = \frac{\boldsymbol{\alpha} e^{\mathbf{A}t}\mathbf{a_k}}{\boldsymbol{\alpha}(-\mathbf{A})^{-1}\mathbf{a_k}}$$

Operations with phase type distributions

Conditional distribution:

$Z = X|X < Y$, where X and Y are independent, X is PH($\alpha, \boldsymbol{A}$) and Y is PH($\beta, \boldsymbol{B}$)

$f_Z^c(t)$ can be obtained from the multi terminal PH distribution

$$\boldsymbol{\gamma} = \boldsymbol{\alpha} \otimes \boldsymbol{\beta},$$

$$\mathbf{G} = \mathbf{A} \oplus \mathbf{B},$$

$$\mathbf{g_a} = \mathbf{a} \oplus \mathbb{1},$$

because:

$$\lim_{\Delta \to 0} \frac{1}{\Delta} Pr(x < X < x + \Delta, X < Y)$$
$$= (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) e^{(\mathbf{A} \oplus \mathbf{B})x} (\mathbf{a} \oplus \mathbb{1})$$

and

$$Pr(X < Y) = (\boldsymbol{\alpha} \otimes \boldsymbol{\beta})(-\mathbf{A} \oplus \mathbf{B})^{-1} (\mathbf{a} \oplus \mathbb{1}),$$

Operations with phase type distributions

Conditional distribution:

$Z = X|X > Y$, where X and Y are independent, X is PH($\alpha, \boldsymbol{A}$) and Y is PH($\beta, \boldsymbol{B}$)

$f_Z^c(t)$ can be obtained from the multi terminal PH distribution

$$\boldsymbol{\gamma} = (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}|\mathbf{0}),$$

$$\mathbf{G} = \left[\begin{array}{c|c} \mathbf{A} \oplus \mathbf{B} & \mathbf{I} \oplus \mathbf{b} \\ \hline \mathbf{0} & \mathbf{A} \end{array}\right],$$

$$\mathbf{g_a} = \left[\begin{array}{c} \mathbf{0} \\ \hline \mathbf{a} \end{array}\right],$$

because:

$$\lim_{\Delta \to 0} \frac{1}{\Delta} Pr(x < X < x + \Delta, X > Y) \\ = (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}|\mathbf{0}) e^{\mathbf{G}x} \mathbf{v}$$

and

$$Pr(X > Y) = (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}|\mathbf{0})(-\mathbf{G})^{-1} \mathbf{v},$$

Properties of phase type distributions

The simplest CPH distribution is the exponential distribution:

$f(t) = \lambda e^{-\lambda t}$, $F(t) = 1 - e^{-\lambda t}$, $f^*(s) = \lambda/(s+\lambda)$

$\mu = \mathbb{E}\tau = 1/\lambda$ and $cv^2 = 1$.

cv^2 is independent of the λ parameter.

The simplest DPH distribution is the geometric distribution:

$$p_k = Pr(X = k) = b_{11}^{k-1}(1 - b_{11}),\ \mathcal{F}(z) = \frac{(1-b_{11})z}{1-b_{11}z}$$

$\mu = \mathbb{E}\tau = 1/(1 - b_{11})$ and $cv^2 = b_{11} = 1 - 1/\mu$.

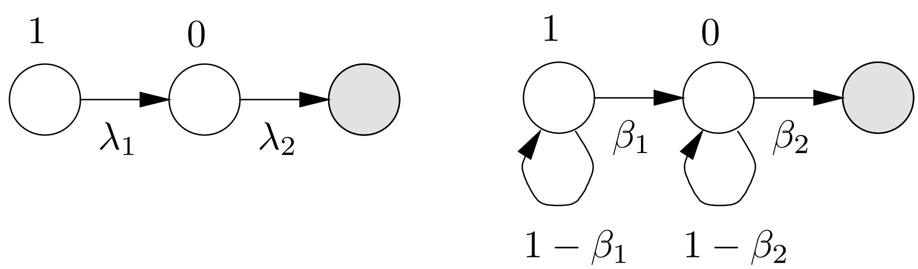
The minimal cv^2 is a function of μ !!!

Properties of phase type distributions

An example:
τ_C and τ_D are *CPH* and *DPH* r.v. with representations $(\boldsymbol{\gamma}, \boldsymbol{\Lambda})$ and $(\boldsymbol{\alpha}, \boldsymbol{B})$, respectively:

$$\boldsymbol{\gamma} = [1, 0] \ , \quad \boldsymbol{\Lambda} = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{bmatrix}$$

$$\boldsymbol{\alpha} = [1, 0] \ , \quad \boldsymbol{B} = \begin{bmatrix} 1-\beta_1 & \beta_1 \\ 0 & 1-\beta_2 \end{bmatrix}$$



$$m_C = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \qquad m_D = \frac{1}{\beta_1} + \frac{1}{\beta_2}$$

$$\sigma_C^2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \qquad \sigma_D^2 = \frac{1}{\beta_1^2} - \frac{1}{\beta_1} + \frac{1}{\beta_2^2} - \frac{1}{\beta_2}$$

$$cv_C^2 = \frac{\lambda_1^2 + \lambda_2^2}{(\lambda_1 + \lambda_2)^2} \quad cv_D^2 = \frac{\beta_1^2 - \beta_1^2\beta_2 + \beta_2^2 - \beta_1\beta_2^2}{(\beta_1 + \beta_2)^2}$$

Properties of phase type distributions

Example 1) Fix λ_1 and β_1 and find λ_2^{min} and β_2^{min} that minimizes cv_C^2 and cv_D^2 :

$$\lambda_2^{min} = \lambda_1 \ ; \quad \beta_2^{min} = \frac{\beta_1(2+\beta_1)}{2-\beta_1} \ .$$

$\rightarrow$ the minimal cv_C^2 is provided by Erlang(2), but the minimal cv_D^2 is not discrete Erlang(2).

Example 2) Fix m_C and m_D, in this case

$$\lambda_1 = \frac{\lambda_2}{m_C\lambda_2 - 1} \ \text{ and } \beta_1 = \frac{\beta_2}{m_D\beta_2 - 1} \ .$$

Find λ_2^{min} and β_2^{min} that minimizes cv_C^2 and cv_D^2 :

$$\lambda_2^{min} = \frac{2}{m_C} \ ; \quad \beta_2^{min} = \frac{2}{m_D}.$$

$\rightarrow$ both cv_C^2 and cv_D^2 are Erlang(2) and the minimal coefficient of variations are:

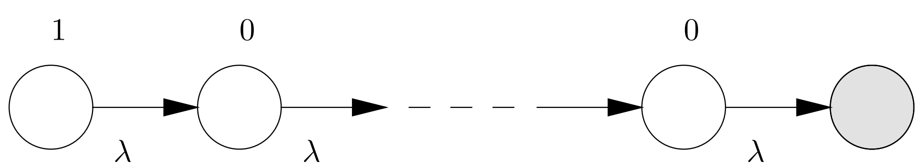
$$cv_C^2 = \frac{1}{2} \text{ and } cv_D^2 = \frac{1}{2} - \frac{1}{m_D} \ .$$

Minimal CV of CPHs

Theorem 1 *The squared coefficient of variation of* τ*,* $cv^2(\tau)$*, satisfies:*

$$cv^2(\tau) \geq \frac{1}{N} \tag{1}$$

and the only CPH distribution, which satisfies the equality is the Erlang(N) distribution:



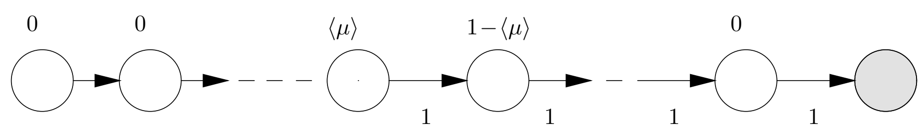
Minimal CV of DPHs

Theorem 2 *The squared coefficient of variation of τ, $cv^2(\tau)$, satisfies the inequality:*

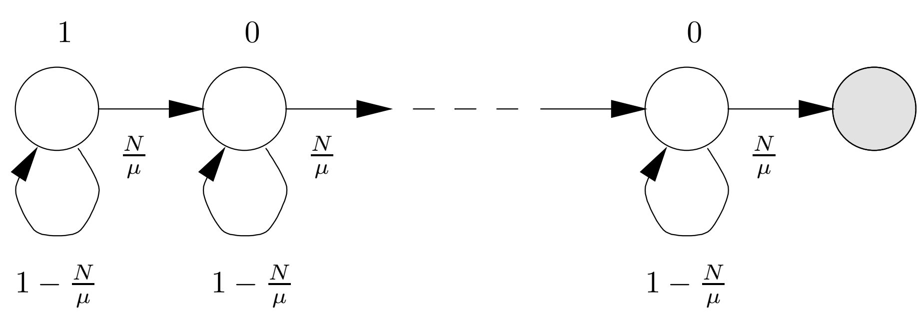
$$cv^2(\tau) \geq \begin{cases} \dfrac{\langle\mu\rangle(1-\langle\mu\rangle)}{\mu^2} & \text{if } \mu < N \ , \\ \\ \dfrac{1}{N} - \dfrac{1}{\mu} & \text{if } \mu \geq N \ . \end{cases} \tag{2}$$

where $\langle x \rangle$ denotes the fraction part of x.

- *for $\mu \leq N$ CV_{min} provided by the mixture of two deterministic distributions, e.g.:*



- *for $\mu > N$ CV_{min} provided by the discrete Erlang distribution:*



Special PH classes

A unique and minimal representation of the PH class is not available yet

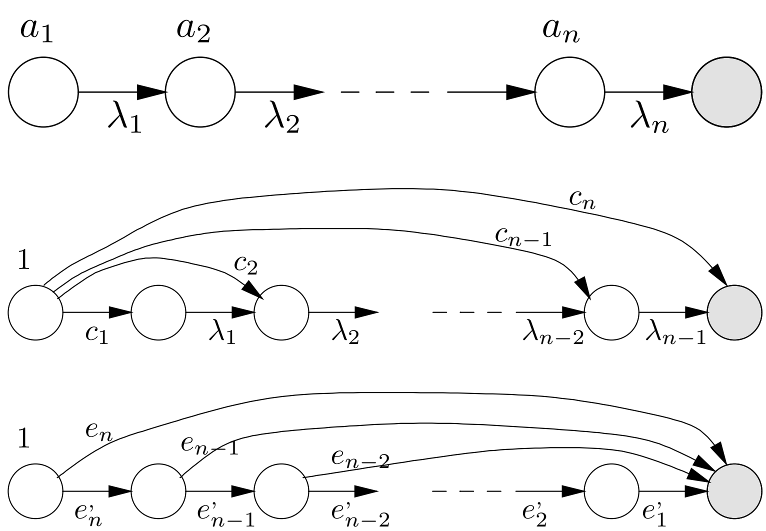
$\rightarrow$ use of simple PH subclasses:

- Acyclic PH distributions
- Hypo-exponential distr. ("series", "$cv < 1$")
- Hyper-exponential distr. ("parallel", "$cv > 1$")
- ...

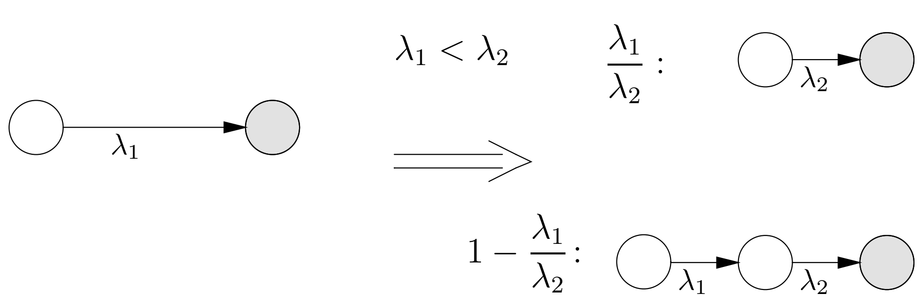
Acyclic PH distributions

The acyclic PH class allows a minimal representation with only $2N$ parameters.

Continuous case: A unique minimal representation of any ACPH distribution is given in one of the three canonical forms:

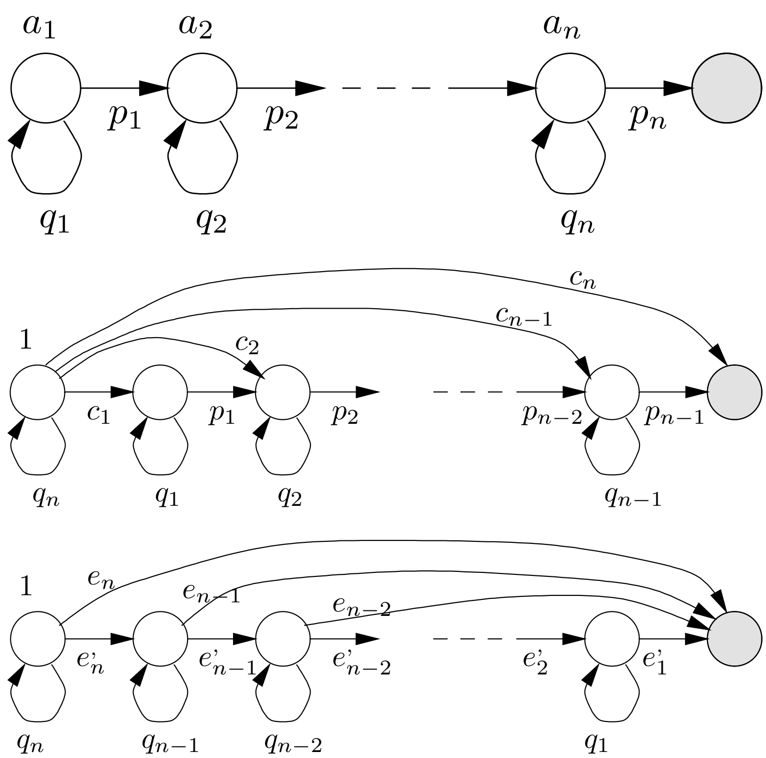


The unique representation is based on the elementary operation:

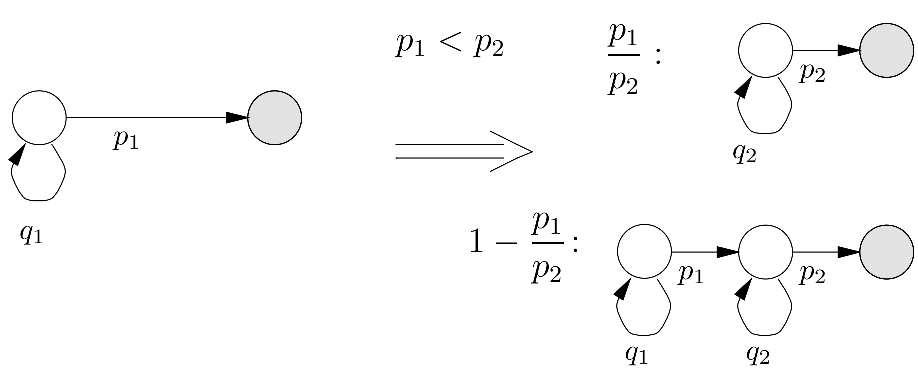


Acyclic PH distributions

Discrete case: A unique minimal representation of any ADPH distribution is given in one of the three canonical forms:



The unique representation is based on the elementary operation:



Fitting with PH distributions

Fitting:
given a non-negative distribution find a "similar" PH distribution.

Formally:

$$\min_{PHparameters} \left\{ \text{Distance}(PH, Original) \right\},$$

where *Distance* is a non-negative valued function.

Measures of similarity:

- a function of a given number of moments (there can be multiple PH distributions with 0 distance)
- a function of the distributions, e.g.,
  - squared CDF difference: $\int_0^\infty (F(t) - \widehat{F}(t))^2 dt$
  - density difference: $\int_0^\infty |f(t) - \widehat{f}(t)| dt$
  - relative entropy: $\int_0^\infty f(t)\ log \left(\frac{f(t)}{\widehat{f}(t)} \right) dt$

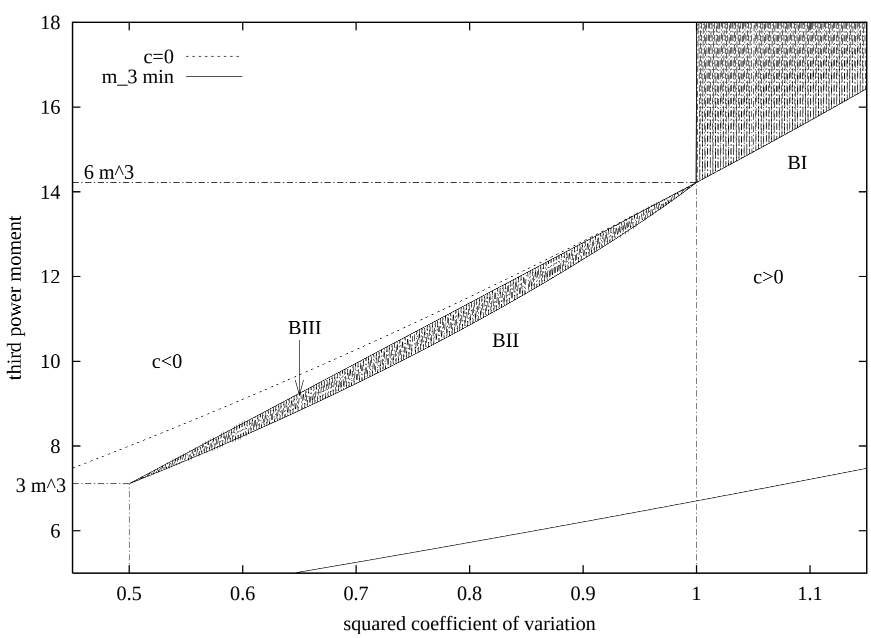
There are also heuristic fitting methods, which are hard to formalize.

Fitting with PH distributions

Moments matching:
Find a PH distribution with the same first K moments.

The PH(N) class has moment limits.
E.g., for an ACPH(2):

- $\mu_1 > 0$
- $\mu_2 > \frac{3}{2}\ \mu_1^2 \qquad (cv^2 > \frac{1}{2})$
- μ_3:



Fitting with PH distributions

Distribution fitting:

Two main approaches:

- EM (expectation maximization) method,
- numerical solution of the non-linear problem:

$$\min_{PHparameters} \left\{ \text{Distance}(PH, Original) \right\}.$$

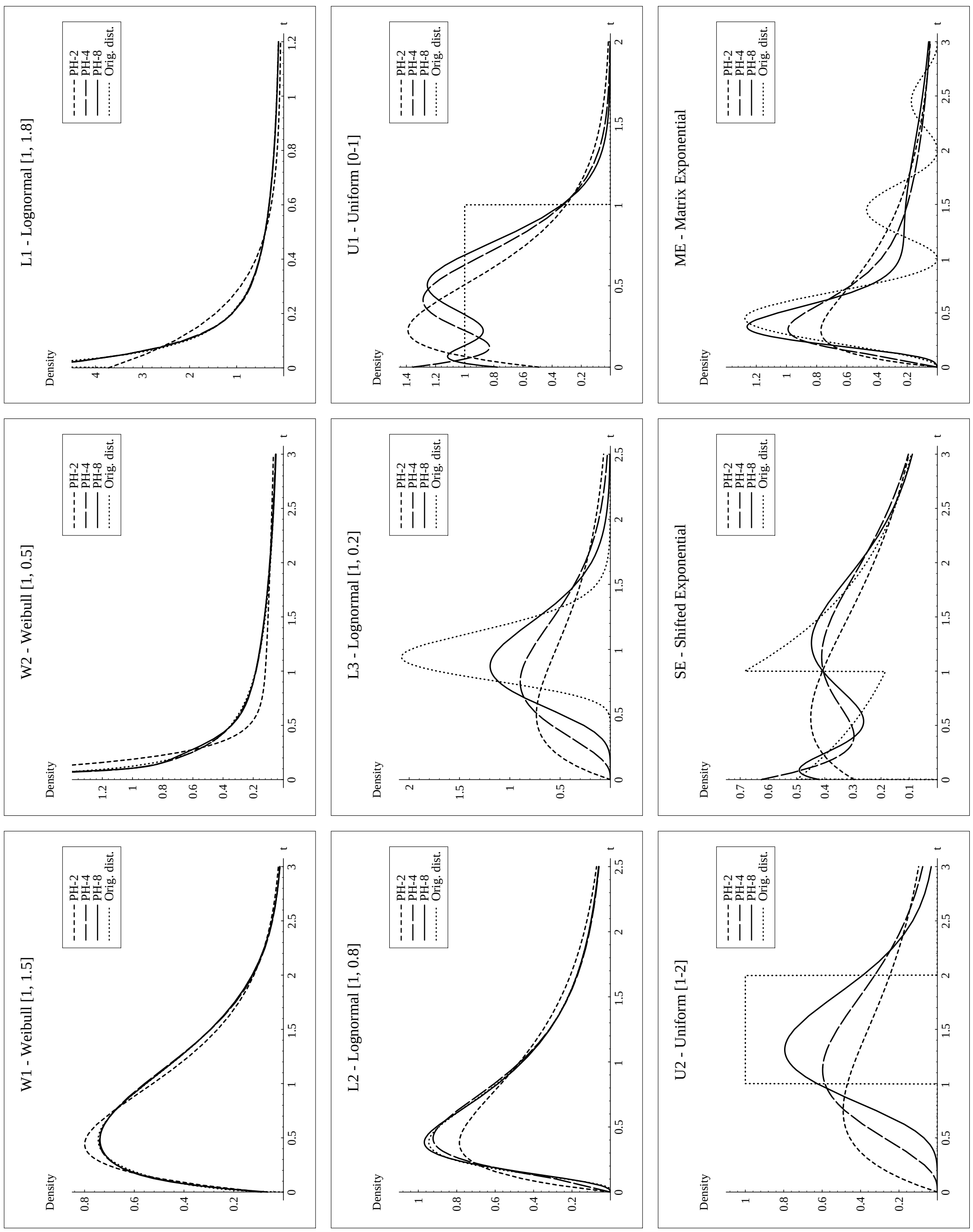
General experiences:

- less PH parameters ($N^2 \to 2N$) $\to$ better fitting ,
- "good" fitting for smooth, mono-mode distributions with light tail.

Problems:

- local minima $\to$ dependence on initial guess,
- numerical instabilities: large N ($\sim$ 10$-$), strange distributions,
- large number of samples.

Fitting with PH distributions



Fitting with PH distributions

Approximating distributions with low coefficient of variation using few phases

$\longrightarrow$ fitting with Discrete PH distributions.

Problems of fitting continuous distributions with discrete PH:

- discretization method
- discrete time step

Fitting with PH distributions

Fitting continuous distributions:

The r.v. X, with *cdf* $F_X(x)$, can be discretized over the discrete set $\mathcal{S} = \{x_1, x_2, x_3, \dots\}$ using, e.g.:

$$x_i = i\ \delta$$

$$p_i = F_X\left(\frac{x_i + x_{i+1}}{2}\right) - F_X\left(\frac{x_{i-1} + x_i}{2}\right)$$

This discretization does not preserve the moments of the distribution.

A natural requirement of discretization is:

$$E(X^i) \sim \delta^i\ E(X_d^i), \qquad i \geq 1\ ,$$

where δ is the discrete time step.

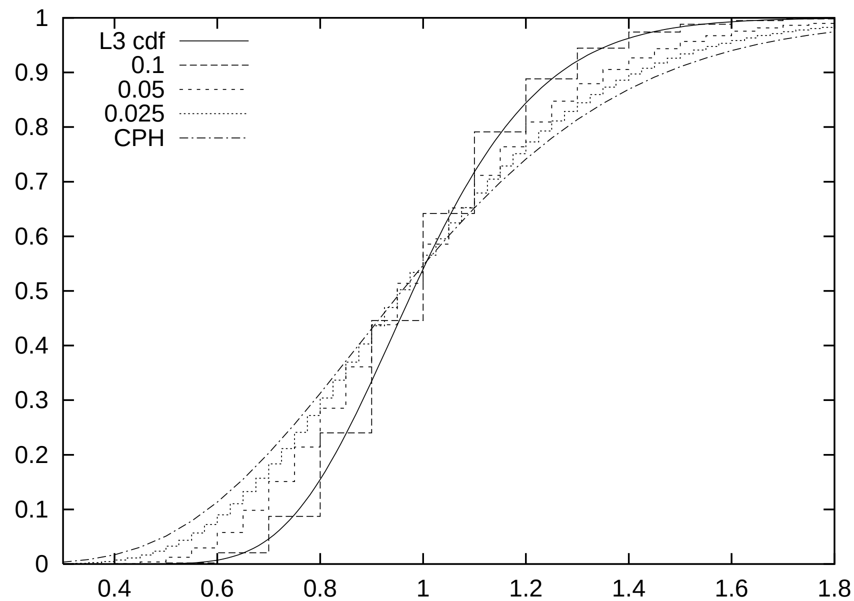
If it is fulfilled

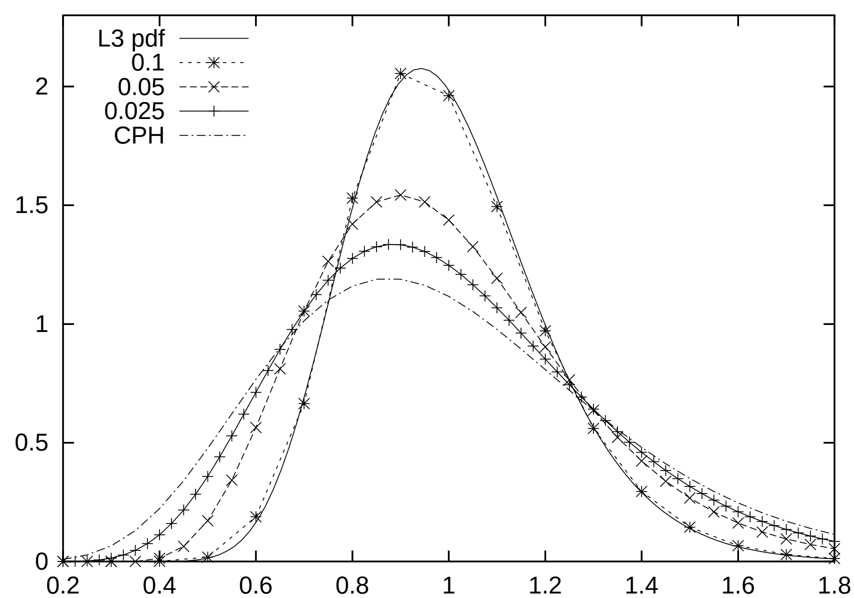
$$E(X) \sim \delta E(X_d) \text{ and } cv(X) \sim cv(X_d)$$

$\rightarrow \delta$ plays significant role in the goodness of fitting.

Fitting with PH distributions

DPHs with different discrete time steps versus CPH





Applications of Phase type distributions

Non-Markovian models $\rightarrow$ Markovian analysis

- queueing models (matrix geometric methods)
- performance, performability models
- stochastic Petri net models

Traditionally continuous time models with CPH were used.

Recently discrete time models gain importance:

- slotted communication protocols
- physical observations at fine time scales
- discrete time stochastic Petri nets
- deterministic or random event time with low variance
- finite support

Matrix exponential/geometric distributions

The continuous distribution with density $f(t)$ is matrix exponential if the Laplace transform of $f(t)$ ($f^*(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$) is a rational function of s.

$$f^*(s) = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_N s^N}{b_0 + b_1 s + b_2 s^2 + \ldots + b_N s^N}$$

The discrete distribution on $\mathbb{N}$ with probability mass function $p_i = Pr(X = i)$ ($i \in \mathbb{N}$) is matrix geometric if the z transform of the probability mass function ($\mathcal{F}(z) = \sum_{i=0}^{\infty} z^i p_i$) is a rational function of z.

$$\mathcal{F}(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_N z^N}{b_0 + b_1 z + b_2 z^2 + \ldots + b_N z^N}$$

The order of a matrix exponential/geometric distribution is the order of the rational function (N).

Properties of matrix exp./geom. distributions

Constraints on the coefficients:

$$f^*(s)|_{s=0} = \mathcal{F}(z)|_{z=1} = 1$$

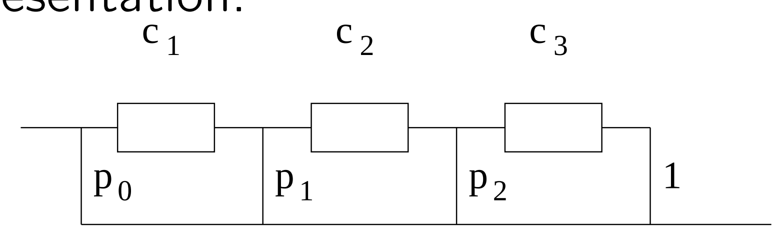
The *poles* of $f^*(s)$ and $\mathcal{F}(z)$ are on the left complex half plane.

Unfortunately, these properties do not ensure a probability distribution.

The set of matrix exp./geom. distributions of order N is a rear subset of the order N rational functions.

Representation of matrix exp./geom. distr.

Cox's representation:



c_i: complex transition rates,
p_i: probability of termination.

"Time domain" representation of matrix exp./geom. distributions:

PDF: $f(t) = \boldsymbol{\alpha} e^{\mathbf{A}t}\mathbf{a}$

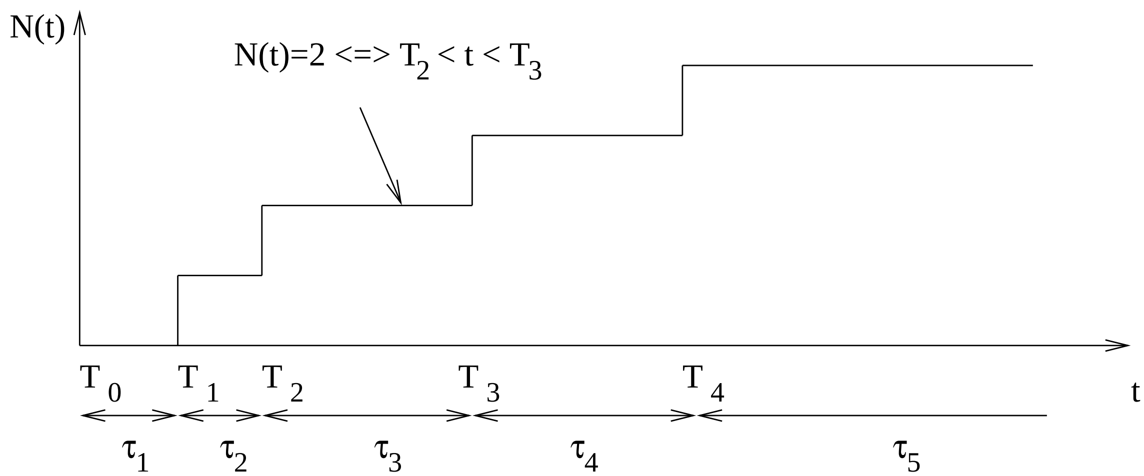
PMF: $p_k = Pr(X = k) = \boldsymbol{\alpha}\mathbf{B}^{k-1}\mathbf{b}$

Where $\alpha, \mathbf{a}, \mathbf{b}$ are general real valued vectors and $\mathbf{A}, \mathbf{B}$ are general real valued square matrices of order N.

Similar to the transform domain representation, only special cases of $\{\alpha, \mathbf{A}, \mathbf{a}\}$ and $\{\alpha, \mathbf{B}, \mathbf{b}\}$ result proper PDFs and PMFs.

Renewal process

Renewal process:
Point/Counting process with i.i.d. inter-event time



Point process: $\tau_1, \tau_2, \tau_3, \ldots$

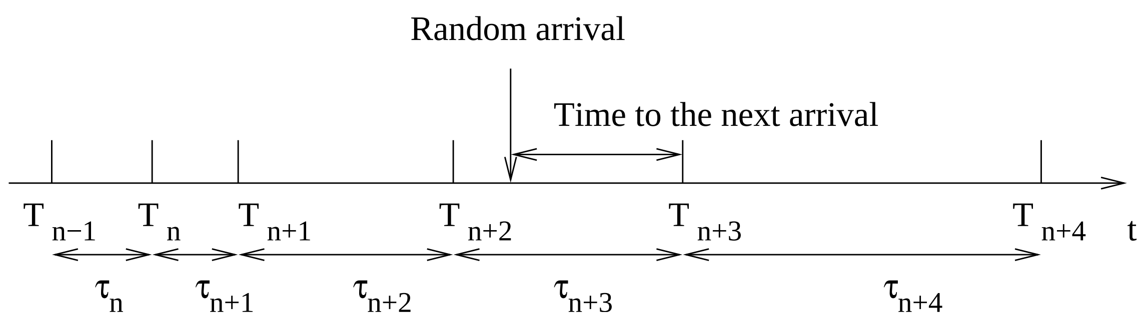
Counting process: $N(t)$

Parameters:

- renewal function: $M(t) = E(N(t))$
- index of dispersion of count: $IDC(t) = \dfrac{\sigma(N(t))}{E(N(t))}$

Equilibrium distribution

Paradox of random arrival in Poisson process.



Paradox:

- τ_n $(n = 1, 2, \ldots)$ is exponentially (λ) distributed,
- due to the memoryless property the time to the next arrival is exponentially (λ) distributed as well.

Is τ_3 longer or the remaining time is shorter than exponential (λ)???

Equilibrium distribution

Basic properties:

- the random arrival instance falls in longer intervals with higher probability,
- inside the selected interval the random arrival instance is uniformly distributed.

Distribution of the length of the selected interval (SI):

$$f_{SI}(t) = \frac{tf(t)}{\int_x xf(x)dx} = \frac{tf(t)}{E(\tau)}$$

Distribution of time to next arrival (T) when the length of the interval is x:

$$f_{T|x}(t) = \begin{cases} 1/x & \text{if } 0 < t < x \\ 0 & \text{otherwise} \end{cases}$$

Equilibrium distribution

Equilibrium distribution:

$$f_T(t) = \int_{x=0}^{\infty} f_{T|x}(t)\ f_{SI}(x)\ dx = \int_{x=t}^{\infty} 1/x\ \frac{xf(x)}{E(\tau)}\ dx$$

$$f_T(t) = \frac{1 - F(t)}{E(\tau)},$$

$$f_T^*(s) = \frac{1/s - F^*(s)}{E(\tau)} = \frac{1 - sF^*(s)}{s\ E(\tau)} = \frac{1 - f^*(s)}{s\ E(\tau)},$$

Moments of equilibrium distribution:

$$E(T^n) = \frac{\int_t t^n(1 - F(t))dt}{E(\tau)} = \frac{E(\tau^{n+1})}{(n+1)E(\tau)}$$

Renewal function

Distribution of T_i:

$$Pr(T_i < t) = \underbrace{F(t) \otimes F(t) \otimes \ldots \otimes F(t)}_{i} = F^{(i)}(t)$$

Renewal function and inter-arrival time distribution:

$$M(t) = E(N(t)) = \sum_{i=0}^{\infty} i \; Pr(N(t) = i) =$$
$$\sum_{i=0}^{\infty} i \; Pr(T_i < t < T_{i+1}) = \sum_{i=0}^{\infty} i \; \Big(Pr(T_i < t) - Pr(T_{i+1} < t) \Big)$$
$$\sum_{i=0}^{\infty} i \; \Big(F^{(i)}(t) - F^{(i+1)}(t) \Big) = \sum_{i=0}^{\infty} i \; F^{(i)}(t) - \sum_{i=1}^{\infty} (i-1) \; F^{(i)}(t)$$

$$\longrightarrow \; M(t) = \sum_{i=1}^{\infty} F^{(i)}(t)$$

Renewal equation

Relation of renewal function and inter-arrival time distribution:

$$
\begin{aligned}
&M(t) = E(N(t)) = F(t) + \sum_{i=1}^{\infty} F^{(i+1)}(t) = \\
&F(t) + \sum_{i=1}^{\infty} \int_{u=0}^{t} F^{(i)}(t-u)\ dF(u) = \\
&F(t) + \int_{u=0}^{t} \sum_{i=1}^{\infty} F^{(i)}(t-u)\ dF(u)
\end{aligned}
$$

$$
\longrightarrow\ \ M(t) = F(t) + \int_{u=0}^{t} M(t-u)\ dF(u)
$$

Renewal equation for densities: $(m(t) = \frac{d}{dt} M(t))$

$$
\longrightarrow\ \ m(t) = f(t) + \int_{u=0}^{t} m(t-u)\ f(u)\ du
$$

Renewal equation in transform domain:

$$
M^{\sim}(s) = \frac{F^{\sim}(s)}{1 - F^{\sim}(s)}, \qquad F^{\sim}(s) = \frac{M^{\sim}(s)}{1 + M^{\sim}(s)}
$$

Initial condition of a renewal process

Depending on the origin of the time access:

- Ordinary renewal process:

  Time starts at renewal $\rightarrow$ the distribution of τ_1 is the same.

- Delayed renewal process:

  Time starts between renewals $\rightarrow$ the distribution of τ_1 is different

$$M^{\sim}(s) = \frac{F_1^{\sim}(s)}{1 - F^{\sim}(s)}$$

- Stationary renewal process:

  (special case) the distribution of τ_1 is the equilibrium distribution

$$M^{\sim}(s) = \frac{F_1^{\sim}(s)}{1 - F^{\sim}(s)} = \frac{\frac{1 - F^{\sim}(s)}{s\ E(T)}}{1 - F^{\sim}(s)} = \frac{1}{s\ E(T)}$$

$$\longrightarrow \quad M(t) = \frac{t}{E(T)}, \quad m(t) = \frac{1}{E(T)}$$

Behaviour of the renewal function

Taylor expansion of $F^{\sim}(s)$:

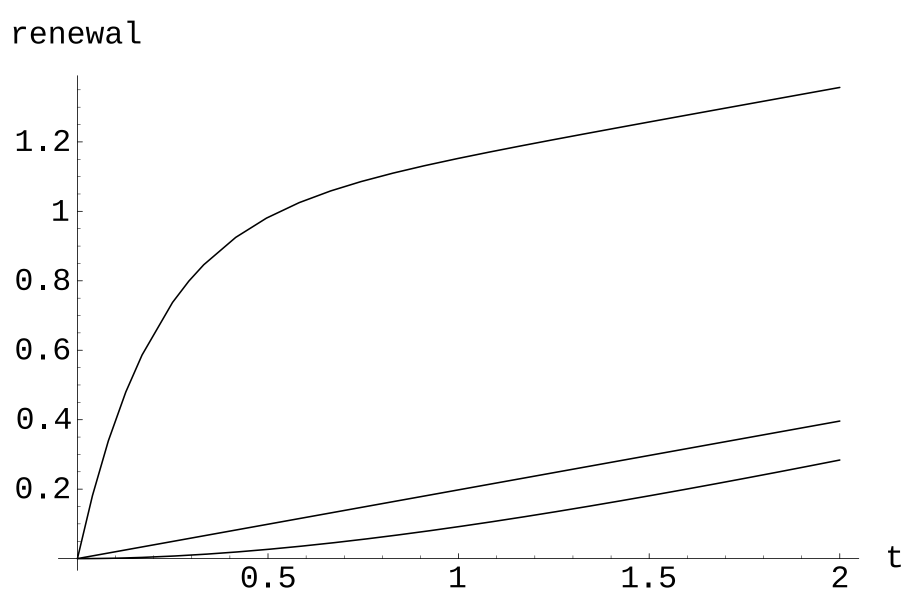
$$F^{\sim}(s) = \sum_{i=0}^{\infty} \frac{s^i}{i!} \left. \frac{d^i}{ds^i} F^{\sim}(s) \right|_{s=0} = \sum_{i=0}^{\infty} (-1)^i \frac{s^i}{i!} \; E(T^i)$$

Series expansion of $M^{\sim}(s)$:

$$M^{\sim}(s) = \frac{F^{\sim}(s)}{1 - F^{\sim}(s)} = \frac{1}{s^2 \; E(T)} + \frac{E(T^2) - 2E^2(T)}{s \; 2E^2(T)} + \sigma\,(1/s)$$

Series expansion of the renewal function:

$$M(t) = \frac{t}{E(T)} + \frac{E(T^2) - 2E^2(T)}{2E^2(T)} + \sigma\,(1)$$

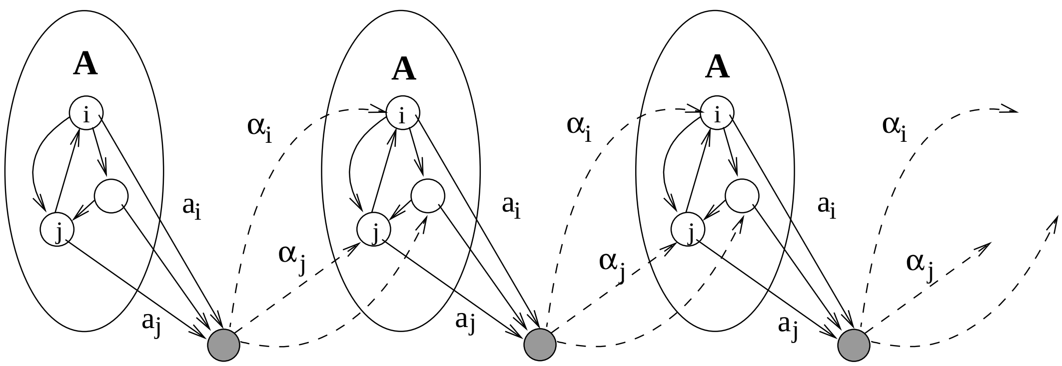


Phase type renewal process

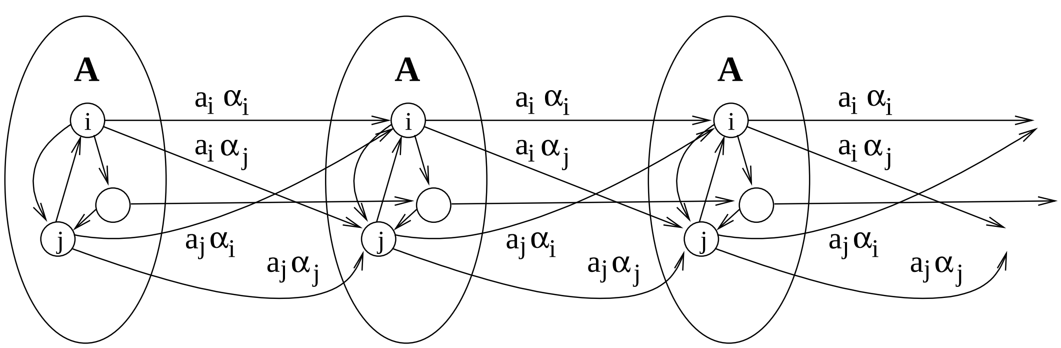
Inter-arrival time is PH distributed.

$N(t)$: number of renewals
$J(t)$: phase of the PH distribution

Logic behaviour:



Tangible behaviour:



$\longrightarrow \{N(t), J(t)\}$ is a Markov chain.

Phase type renewal process

Structure of the generator matrix:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|}\hline \mathbf{A} & \mathbf{a}\boldsymbol{\alpha} & & & \\ \hline & \mathbf{A} & \mathbf{a}\boldsymbol{\alpha} & & \\ \hline & & \mathbf{A} & \mathbf{a}\boldsymbol{\alpha} & \\ \hline & & & \mathbf{A} & \mathbf{a}\boldsymbol{\alpha} \\ \hline & & & & \ddots \\ \hline \end{array}$$

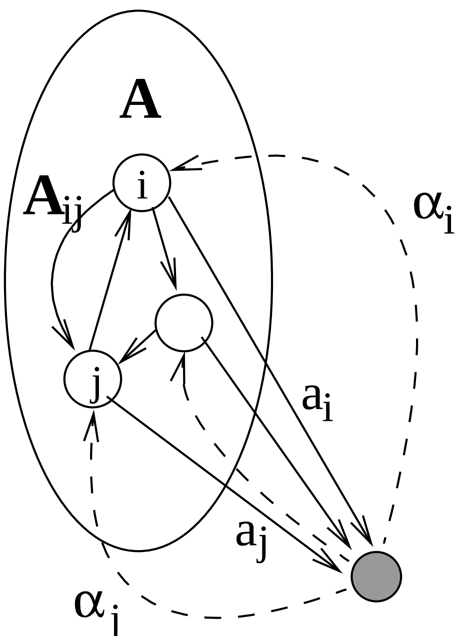
On the block level it is similar to the structure of a Poisson process.

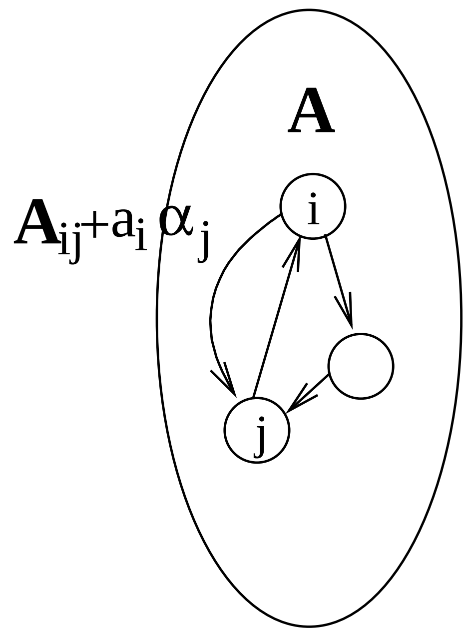
$\longrightarrow$ "quasi" birth process.

Phase type renewal process

Phase process $(J(t))$ is a CTMC:

Logic behaviour: Tangible behaviour:





Generator matrix: $\mathbf{Q_J} = \mathbf{A} + \mathbf{a}\boldsymbol{\alpha}$

Properties:

- for $i \neq j$: $\mathbf{Q_J}_{ij} = \mathbf{A}_{ij} + \mathbf{a}_i \boldsymbol{\alpha}_j \geq 0$,
- $\mathbf{Q_J}\mathbb{1} = \mathbf{A}\mathbb{1} + \mathbf{a}\underbrace{\boldsymbol{\alpha}\mathbb{1}}_{1} = \mathbf{A}\mathbb{1} + \mathbf{a} = \mathbf{0}$

Phase type renewal process

$M(t) = E(N(t))$, $m(t) = \frac{d}{dt}M(t)$

Short time behaviour:

$$\{N(t+\Delta) - N(t) | J(t) = i\} = \begin{cases} 0 & 1 - a_i\Delta + \sigma(\Delta) \\ 1 & a_i\Delta + \sigma(\Delta) \\ > 1 & \sigma(\Delta) \end{cases}$$

$\longrightarrow$

$$E(N(t+\Delta) - N(t) | J(t) = i) =$$
$$\{M(t+\Delta) - M(t) | J(t) = i\} = a_i\Delta + \sigma(\Delta)$$

$\longrightarrow$

$$\{m(t) | J(t) = i\} = a_i$$

$\longrightarrow$

$$m(t) = \sum_{i \in S} Pr(J(t) = i)\ a_i =$$
$$\sum_{j \in S} \sum_{i \in S} \alpha_j Pr(J(t) = i | J(0) = j)\ a_i =$$
$$\sum_{j \in S} \sum_{i \in S} \alpha_j \left[e^{\mathbf{Q_J} t}\right]_{ji} a_i = \boldsymbol{\alpha}\ e^{\mathbf{Q_J} t}\ \mathbf{a}$$

Phase type renewal process

The time to the next arrival at an arbitrary time t is PH distributed with parameters $(\mathbf{p}(t), \mathbf{A})$,

where $\mathbf{p}(t)$ is the transient distribution of the phase process $(p_i(t) = Pr(J(t) = i))$

Initial condition of a PH renewal process:

- Ordinary PH renewal process:

  $\mathbf{p}(0) = \boldsymbol{\alpha}$,

- Delayed PH renewal process:

  $\mathbf{p}(0)$ is an arbitrary distribution,

- Stationary PH renewal process:

  $\mathbf{p}(0) = \boldsymbol{\pi}$,

  where $\boldsymbol{\pi}$ is the stationary distribution of the phase process. $(\mathbf{0} = \boldsymbol{\pi}\mathbf{Q_J}, \boldsymbol{\pi}\mathbb{1} = 1)$

Phase type renewal process

Sojourn time in state j in a renewal interval starting from state i: T_{ij}.

Mean sojourn time:

$$E(T_{ij}) = \frac{\delta_{ij}}{-A_{ii}} + \sum_{k,k\neq i} \frac{A_{ik}}{-A_{ii}} E(T_{kj})$$

$$0 = \delta_{ij} + \sum_{k} A_{ik} E(T_{kj}) \quad \longrightarrow \quad \mathbf{0} = \mathbf{I} + \mathbf{A}\overline{\mathbf{T}}$$

$$\overline{\mathbf{T}} = (-\mathbf{A})^{-1} \quad \longrightarrow \quad E(T_{ij}) = [(-\mathbf{A})^{-1}]_{ij}$$

Phase type renewal process

Sojourn time distribution, $i = j$:

$$t^*_{ii}(s) = \frac{-A_{ii}}{s - A_{ii}} \left(\frac{a_i}{-A_{ii}} + \sum_{k, k \neq i} \frac{A_{ik}}{-A_{ii}} t^*_{ki}(s) \right)$$

$$s \; t^*_{ii}(s) = a_i + \sum_k A_{ik} t^*_{ki}(s)$$

$i \neq j$:

$$t^*_{ij}(s) = \frac{a_i}{-A_{ii}} + \sum_{k, k \neq i} \frac{A_{ik}}{-A_{ii}} t^*_{kj}(s)$$

$$0 = a_i + \sum_k A_{ik} t^*_{kj}(s)$$

in general:

$$\delta_{ij} \;\; s \;\; t^*_{ij}(s) = a_i + \sum_k A_{ik} t^*_{kj}(s)$$

$$s \;\; \mathrm{Diag}\langle t^*_{ii}(s) \rangle = \mathbf{a}\mathbf{e} + \mathbf{A} \;\; \mathbf{t}^*(s),$$

where $\mathbf{e} = \{1, 1, \ldots, 1\}$.

Phase type renewal process

Mean time spent in j in a renewal interval:

$$\tau_j = \sum_i \alpha_i E(T_{ij}) \quad \longrightarrow \quad \boldsymbol{\tau} = \boldsymbol{\alpha}(-\mathbf{A})^{-1}$$

Portion of time spent in j in a renewal interval:

$$\nu_j = \frac{\tau_j}{\sum_k \tau_k} = \frac{\sum_i \alpha_i E(T_{ij})}{\sum_k \sum_i \alpha_i E(T_{ik})} \quad \longrightarrow \quad \boldsymbol{\nu} = \frac{\boldsymbol{\alpha}(-\mathbf{A})^{-1}}{\boldsymbol{\alpha}(-\mathbf{A})^{-1}\mathbb{1}}$$

Theorem: $\boldsymbol{\nu} = \boldsymbol{\pi}$ (time average = stationary behaviour)

Proof: $\boldsymbol{\nu}\mathbb{1} = 1$ by definition, and

$$\boldsymbol{\nu}\mathbf{Q_j} = \boldsymbol{\nu}(\mathbf{A} + \mathbf{a}\boldsymbol{\alpha}) = \frac{\boldsymbol{\alpha}(-\mathbf{A})^{-1}\ (\mathbf{A} - \mathbf{A}\mathbb{1}\boldsymbol{\alpha})}{\boldsymbol{\alpha}(-\mathbf{A})^{-1}\mathbb{1}}$$

$$= \frac{-\boldsymbol{\alpha} + \boldsymbol{\alpha}\mathbb{1}\boldsymbol{\alpha}}{\boldsymbol{\alpha}(-\mathbf{A})^{-1}\mathbb{1}} = \frac{-\boldsymbol{\alpha} + \boldsymbol{\alpha}}{\boldsymbol{\alpha}(-\mathbf{A})^{-1}\mathbb{1}} = \mathbf{0}$$

Phase type renewal process

If $\mathbf{Q}$ is a generator of an irreducible CTMC then $\mathbf{Q}$ is singular ($\nexists \mathbf{Q}^{-1}$), since 0 is an eigenvalue of $\mathbf{Q}$ ($\mathbf{0} = \boldsymbol{\pi}\mathbf{Q}$).

Theorem: $(\mathbf{Q} - \mathbb{1}\boldsymbol{\pi})$ is non-singular.
(I.e. $(\mathbf{Q} - \mathbb{1}\boldsymbol{\pi})^{-1}$ exists.)

Proof: Assume $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{x}(\mathbf{Q} - \mathbb{1}\boldsymbol{\pi}) = \mathbf{0}$,

then $\mathbf{x}(\mathbf{Q} - \mathbb{1}\boldsymbol{\pi})\mathbb{1} = \mathbf{0}\mathbb{1} \longrightarrow \mathbf{x}\mathbb{1} = 0$;

but from $\mathbf{x}(\mathbf{Q} - \mathbb{1}\boldsymbol{\pi}) = \mathbf{0}$ we also have $\mathbf{x}\mathbf{Q} = \underbrace{\mathbf{x}\mathbb{1}}_{0}\boldsymbol{\pi} = \mathbf{0}$,

it is possible only when $\mathbf{x}$ is proportional to $\boldsymbol{\pi}$, but it is in contrast with $\mathbf{x}\mathbb{1} = 0$.

Phase type renewal process

Analysis of the renewal function, $M(t) = E(N(t))$:

$$M(t) = \int_{x=0}^{t} m(x)\ dx = \boldsymbol{\alpha} \int_{x=0}^{t} e^{\mathbf{Q_J} x}\ dx\ \mathbf{a} =$$

$$\boldsymbol{\alpha} \int_{x=0}^{t} \sum_{i=0}^{\infty} \frac{x^i}{i!}\ \mathbf{Q_J}^i\ dx\ \mathbf{a} = \boldsymbol{\alpha} \sum_{i=0}^{\infty} \frac{t^{i+1}}{(i+1)!}\ \mathbf{Q_J}^i\ \mathbf{a} =$$

$$\boldsymbol{\alpha} \sum_{i=0}^{\infty} \frac{t^{i+1}}{(i+1)!}\ \mathbf{Q_J}^i (\mathbf{Q_J} - \mathbb{1}\boldsymbol{\pi})(\mathbf{Q_J} - \mathbb{1}\boldsymbol{\pi})^{-1}\ \mathbf{a} =$$

$$\boldsymbol{\alpha} \sum_{i=0}^{\infty} \frac{t^{i+1}}{(i+1)!}\ \mathbf{Q_J}^{i+1} (\mathbf{Q_J} - \mathbb{1}\boldsymbol{\pi})^{-1}\ \mathbf{a}$$

$$- \boldsymbol{\alpha} \sum_{i=0}^{\infty} \frac{t^{i+1}}{(i+1)!}\ \underbrace{\mathbf{Q_J}^i \mathbb{1}}_{0 \text{ if } i>0} \boldsymbol{\pi} (\mathbf{Q_J} - \mathbb{1}\boldsymbol{\pi})^{-1}\ \mathbf{a} =$$

$$\boldsymbol{\alpha}(e^{\mathbf{Q_J} t} - \mathbf{I})(\mathbf{Q_J} - \mathbb{1}\boldsymbol{\pi})^{-1}\ \mathbf{a} - t\ \underbrace{\boldsymbol{\alpha}\mathbb{1}}_{1} \underbrace{\boldsymbol{\pi}(\mathbf{Q_J} - \mathbb{1}\boldsymbol{\pi})^{-1}}_{-\boldsymbol{\pi}}\ \mathbf{a} =$$

$$\boldsymbol{\alpha}(e^{\mathbf{Q_J} t} - \mathbf{I})(\mathbf{Q_J} - \mathbb{1}\boldsymbol{\pi})^{-1}\ \mathbf{a} + t\ \boldsymbol{\pi}\ \mathbf{a}$$

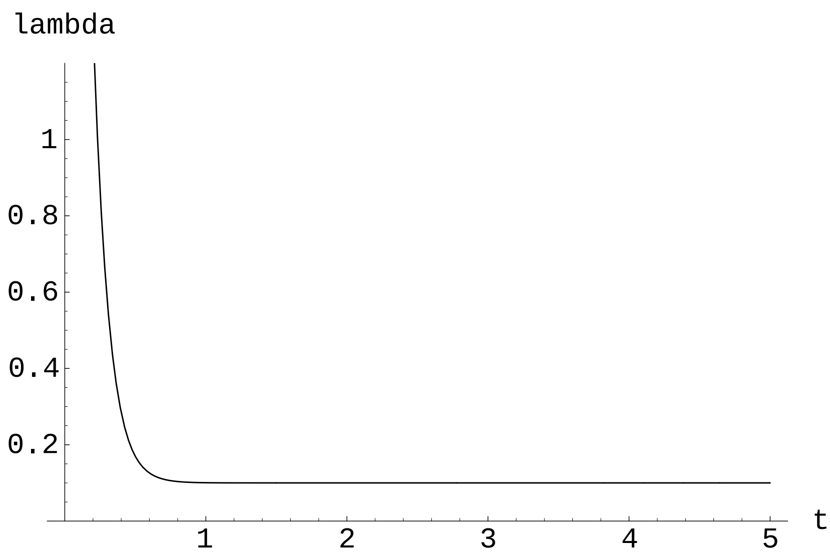
since

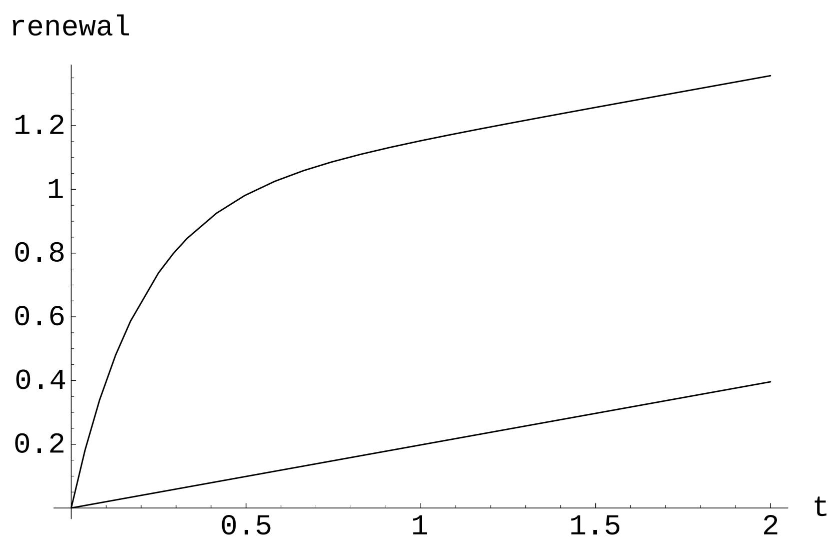
$$\boldsymbol{\pi} = \underbrace{\boldsymbol{\pi}\mathbb{1}}_{1} \boldsymbol{\pi} - \underbrace{\boldsymbol{\pi}\mathbf{Q_J}}_{\mathbf{0}} = \boldsymbol{\pi}(\mathbb{1}\boldsymbol{\pi} - \mathbf{Q_J})$$

Phase type renewal process

Examples:

hyper-exponential: $\boldsymbol{\alpha} = \{0.5, 0.5\}$, $\mathbf{A} = \begin{array}{|c|c|} \hline -0.1 & 0 \\ \hline 0 & -10 \\ \hline \end{array}$

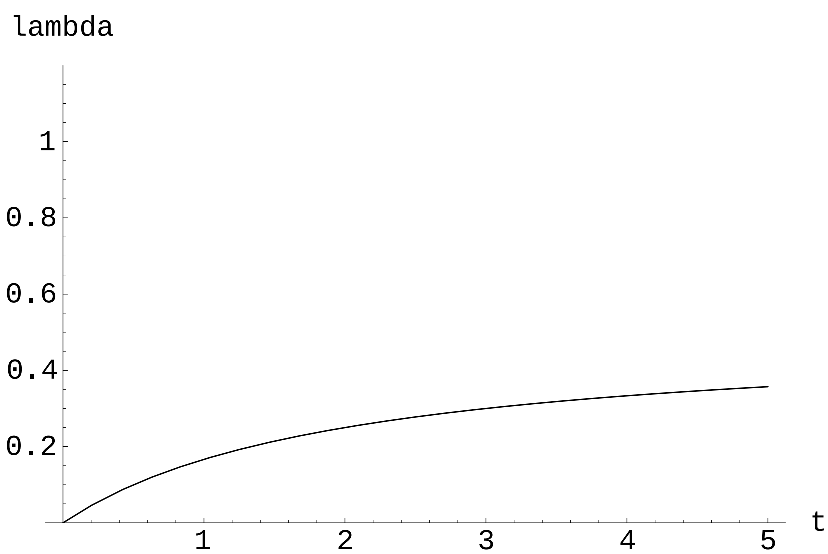


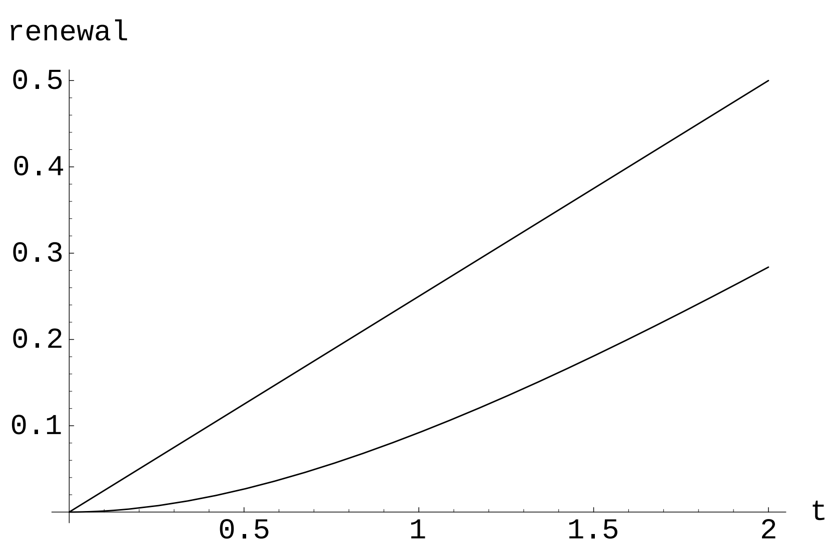


Phase type renewal process

Examples:

hypo-exponential: $\boldsymbol{\alpha} = \{1, 0\}$, $\mathbf{A} = \begin{array}{|c|c|}\hline -0.5 & 0.5 \\ \hline 0 & -0.5 \\ \hline \end{array}$





Phase type renewal process

Distribution of the number of renewals
(short term behaviour)

$$P_j(n,t) = Pr(N(t) = n, J(t) = j),$$

$$\tilde{\mathbf{P}}(n,t) = \{P_j(n,t)\} \quad \text{(vector)}$$

The transient behaviour of the $(N(t), J(t))$ CTMC is $\dfrac{d\tilde{\mathbf{P}}(t)}{dt} = \tilde{\mathbf{P}}(t)\mathbf{Q}$, where $\tilde{\mathbf{P}}(t) = \{\tilde{\mathbf{P}}(0,t), \tilde{\mathbf{P}}(1,t), \ldots\}$.

For $\tilde{\mathbf{P}}(0,t)$ and $\tilde{\mathbf{P}}(i,t)$ $(i > 0)$ we have:

$$\frac{d\tilde{\mathbf{P}}(0,t)}{dt} = \tilde{\mathbf{P}}(0,t)\mathbf{A}$$

$$\frac{d\tilde{\mathbf{P}}(i,t)}{dt} = \tilde{\mathbf{P}}(i,t)\mathbf{A} + \tilde{\mathbf{P}}(i-1,t)\mathbf{a}\boldsymbol{\alpha}$$

with initial conditions: $\tilde{\mathbf{P}}(0,0) = \boldsymbol{\alpha}$, $\tilde{\mathbf{P}}(i,0) = \mathbf{0}$.

z-transform:

$$\frac{d\hat{\mathbf{P}}(z,t)}{dt} = \hat{\mathbf{P}}(z,t)\mathbf{A} + z\hat{\mathbf{P}}(z,t)\mathbf{a}\boldsymbol{\alpha} = \hat{\mathbf{P}}(z,t)(\mathbf{A} + z\mathbf{a}\boldsymbol{\alpha})$$

with initial condition: $\hat{\mathbf{P}}(z,0) = \boldsymbol{\alpha}$.

Solution: $\hat{\mathbf{P}}(z,t) = \boldsymbol{\alpha}\ e^{(\mathbf{A}+z\mathbf{a}\boldsymbol{\alpha})t}$

Phase type renewal process

Distribution of the number of renewals
(renewal theory)

$$P_{ij}(n,t) = Pr(N(t) = n, J(t) = j | J(0) = i)$$
$$\mathbf{P}(n,t) = \{P_{ij}(n,t)\} \quad \text{(matrix)}$$

no renewal in $(0,t)$: $\mathbf{P}(0,t) = e^{\mathbf{A}t}$

renewal in $(0,t)$ (at time $t-u$):

$$\mathbf{P}(k,t) = \int_{u=0}^{t} e^{\mathbf{A}(t-u)}\mathbf{a}\boldsymbol{\alpha}\mathbf{P}(k-1,u)du$$

z-transform:

$$\mathbf{P}(z,t) = e^{\mathbf{A}t} + z\int_{u=0}^{t} e^{\mathbf{A}(t-u)}\mathbf{a}\boldsymbol{\alpha}\mathbf{P}(z,u)du$$

Solution can be obtained by multiplying with $e^{\mathbf{A}t}$ and calculating derivatives.

Markov arrival process

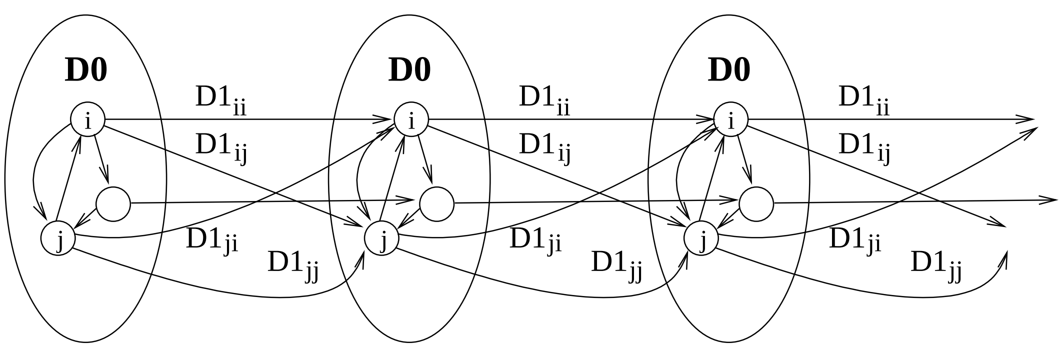
A point process characterized by

- $N(t)$: number of arrivals
- $J(t)$: phase of the PH distribution

Restriction in PH renewal process:
  the phase distribution is reset at arrivals.

MAP:
  the phase distribution after an arrival is arbitrary.

Process behaviour:



$\longrightarrow \{N(t), J(t)\}$ is still a Markov chain.

Markov arrival process

Common notation:

- $\mathbf{D_0} = A$ – phase transitions without arrival
- $\mathbf{D_1}$ – phase transitions with one arrival

Structure of the generator matrix:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|} \hline \mathbf{D_0} & \mathbf{D_1} & & & \\ \hline & \mathbf{D_0} & \mathbf{D_1} & & \\ \hline & & \mathbf{D_0} & \mathbf{D_1} & \\ \hline & & & \mathbf{D_0} & \mathbf{D_1} \\ \hline & & & & \ddots \\ \hline \end{array}$$

On the block level it is similar to the structure of a Poisson process.

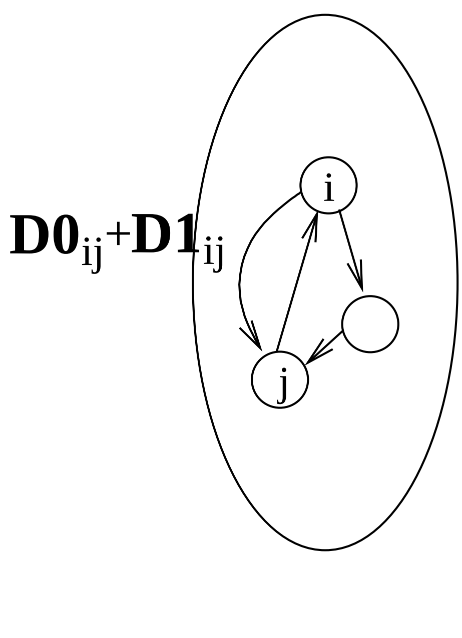
$\longrightarrow$ "quasi" birth process.

Properties of Markov arrival process

- the phase distribution at arrival instances form a DTMC with $\mathbf{P} = (-\mathbf{D_0})^{-1}\mathbf{D_1}$
  $\longrightarrow$ correlated initial phase distributions,

- inter-arrival time is PH distributed with representation $(\boldsymbol{\alpha}_0, \mathbf{D_0})$, $(\boldsymbol{\alpha}_1, \mathbf{D_0})$, $(\boldsymbol{\alpha}_2, \mathbf{D_0})$, ...
  $\longrightarrow$ correlated inter-arrival times,

- phase process $(J(t))$ is a CTMC with generator $\mathbf{D} = \mathbf{D_0} + \mathbf{D_1}$



Properties of Markov arrival process

- (time) stationary phase distribution α is the solution of $\alpha\mathbf{D} = 0, \alpha\mathbb{1} = 1$.

- (embedded) stationary phase distribution after an arrival π is the solution of $\pi\mathbf{P} = \pi, \pi\mathbb{1} = 1$.

- stationary inter arrival time (X) is PH distributed with representation $(\pi, \mathbf{D_0})$, whose nth moment is $E(X^n) = n!\pi(-\mathbf{D_0})^{-n}\mathbb{1}$.

- the initial and consecutive state dependent density of the inter arrival time is $[f_{ij}(t)] = e^{\mathbf{D_0}t}\mathbf{D_1}$.

- the stationary arrival intensity is
$\lambda = \alpha\mathbf{D_1}\mathbb{1} = \dfrac{1}{E(X)} = \dfrac{1}{\pi(-\mathbf{D_0})^{-1}\mathbb{1}}$.

- similar to PH renewal processes
$\alpha = \dfrac{\pi(-\mathbf{D_0})^{-1}}{\pi(-\mathbf{D_0})^{-1}\mathbb{1}} = \dfrac{\pi(-\mathbf{D_0})^{-1}}{E(X)} = \lambda\pi(-\mathbf{D_0})^{-1}$.

Properties of Markov arrival process

The joint pdf of X_0 and X_k is

$$f_{X_0,X_k}(x,y) = \pi e^{\mathbf{D_0}x}\mathbf{D_1}\mathbf{P}^{k-1}e^{\mathbf{D_0}y}\mathbf{D_1}\mathbb{1}.$$

Due to the Markovian behaviour of MAPs X_0 and X_k depend only via their initial states !!!!

Properties of Markov arrival process

Lag k correlation:

$$E(X_0 X_k) = \int_{t=0}^{\infty} \int_{\tau=0}^{\infty} t \ \tau \ \pi e^{\mathbf{D_0} t} \mathbf{D_1} \mathbf{P}^{k-1} e^{\mathbf{D_0} \tau} \mathbf{D_1} \mathbb{1} \ d\tau \ dt$$

$$= \pi (-\mathbf{D_0})^{-2} \mathbf{D_1} \mathbf{P}^{k-1} (-\mathbf{D_0})^{-2} \underbrace{\mathbf{D_1} \mathbb{1}}_{-\mathbf{D_0} \mathbb{1}}$$

$$= \pi (-\mathbf{D_0})^{-1} \mathbf{P}^{k} (-\mathbf{D_0})^{-1} \mathbb{1} = \frac{1}{\lambda} \alpha \mathbf{P}^{k} (-\mathbf{D_0})^{-1} \mathbb{1}$$

Since

$$\int_{t=0}^{\infty} t \ e^{\mathbf{D_0} t} dt = \underbrace{\left[t \ (\mathbf{D_0})^{-1} e^{\mathbf{D_0} t} \right]_0^{\infty}}_{0} - \int_{t=0}^{\infty} (\mathbf{D_0})^{-1} \ e^{\mathbf{D_0} t} dt$$

and

$$\int_{t=0}^{\infty} e^{\mathbf{D_0} t} dt = \lim_{T \to \infty} \sum_{i=0}^{\infty} \frac{\mathbf{D_0}^i}{i!} \int_0^T t^i dt = \lim_{T \to \infty} \sum_{i=0}^{\infty} \frac{\mathbf{D_0}^i}{i!} \frac{T^{i+1}}{i+1}$$

$$= \lim_{T \to \infty} (\mathbf{D_0})^{-1} \Bigg(\underbrace{e^{\mathbf{D_0} T}}_{\to 0} - I \Bigg) = (-\mathbf{D_0})^{-1}$$

Properties of Markov arrival process

Covariance:

$$Cov(X_0, X_k) = E(X_0 X_k) - E^2(X) =$$

$$= \frac{1}{\lambda}\ \alpha \mathbf{P}^k (-\mathbf{D_0})^{-1} \mathbb{1} - \frac{1}{\lambda^2}$$

Coefficient of correlation:

$$Corr(X_0, X_k) = \frac{Cov(X_0, X_k)}{E(X^2) - E^2(X)} = \frac{\frac{E(X_0 X_k)}{E^2(X)} - 1}{\frac{E(X^2)}{E^2(X)} - 1}$$

$$= \frac{\lambda\ \alpha \mathbf{P}^k (-\mathbf{D_0})^{-1} \mathbb{1} - 1}{2\lambda\ \alpha (-\mathbf{D_0})^{-1} \mathbb{1} - 1}$$

Properties of Markov arrival process

In general, for $a_0 = 0 < a_1 < a_2 < \ldots < a_k$,

the joint density is:

$$f_{X_{a_0},X_{a_1},\ldots,X_{a_k}}(x_0, x_1, \ldots, x_k) =$$

$$= \pi e^{\mathbf{D_0}x_0}\mathbf{D_1}\mathbf{P}^{a_1-a_0-1}e^{\mathbf{D_0}x_1}\mathbf{D_1}\mathbf{P}^{a_2-a_1-1}\ldots e^{\mathbf{D_0}x_k}\mathbf{D_1}\mathbb{1} \ ,$$

and the joint moment is:

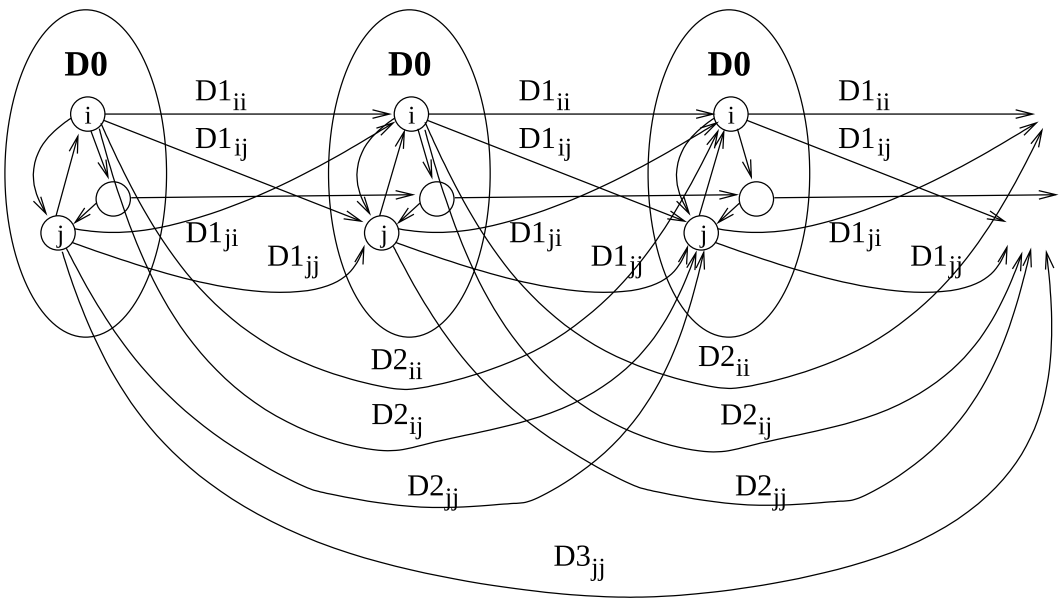
$$E(X_{a_0}^{i_0}, X_{a_1}^{i_0}, \ldots, X_{a_k}^{i_0}) =$$

$$= \pi i_0!(-\mathbf{D_0})^{-i_0}\mathbf{P}^{a_1-a_0} i_1!(-\mathbf{D_0})^{-i_1}\mathbf{P}^{a_2-a_1}\ldots i_k!(-\mathbf{D_0})^{-i_k}\mathbb{1} \ .$$

Batch Markov arrival process

MAP with batch arrivals.

Process behaviour:



$\longrightarrow \{N(t), J(t)\}$ is still a Markov chain.

Batch Markov arrival process

Common notation:

- $\mathbf{D_0}$ – phase transitions without arrival
- $\mathbf{D_k}$ – phase transitions with k arrivals

Structure of the generator matrix:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|}
\hline
\mathbf{D_0} & \mathbf{D_1} & \mathbf{D_2} & \mathbf{D_3} & \mathbf{D_4} \\
\hline
 & \mathbf{D_0} & \mathbf{D_1} & \mathbf{D_2} & \mathbf{D_3} \\
\hline
 & & \mathbf{D_0} & \mathbf{D_1} & \mathbf{D_2} \\
\hline
 & & & \mathbf{D_0} & \mathbf{D_1} \\
\hline
 & & & & \ddots \\
\hline
\end{array}$$

Properties of matrices $\mathbf{D_k}$:

- $\mathbf{D_0}$: $\mathbf{D_0}_{ij} \geq 0$ for $i \neq j$, and $\mathbf{D_0}_{ii} \leq 0$
- for $k \geq 1$: $\mathbf{D_k}_{ij} \geq 0$
- $\displaystyle\sum_{k=0}^{\infty} \mathbf{D_k} \mathbb{1} = \mathbf{0}$ (row-sum=0)

Batch Markov arrival process

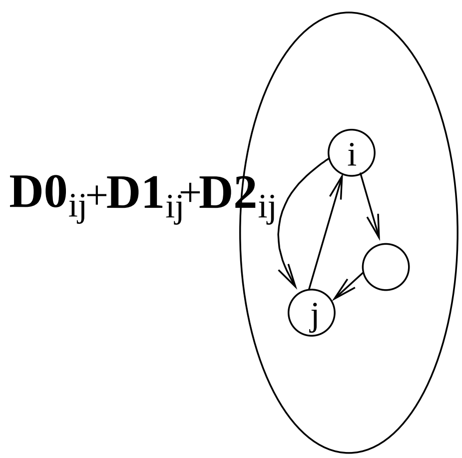
Properties of batch Markov arrival process:

- the phase distribution at arrival instances form a DTMC
  $\longrightarrow$ correlated initial phase distributions,

- inter-arrival time is PH distributed with representation $(\boldsymbol{\alpha}_0, \mathbf{D_0})$, $(\boldsymbol{\alpha}_1, \mathbf{D_0})$, $(\boldsymbol{\alpha}_2, \mathbf{D_0})$, ...
  $\longrightarrow$ correlated inter-arrival times,

- batch arrivals,

- phase process $(J(t))$ is a CTMC with generator $\mathbf{D} = \sum_{k=0}^{\infty} \mathbf{D_k}$



Batch Markov arrival process

Examples:

- bath PH renewal process:
  $\mathbf{D_0} = \mathbf{A}$, $\mathbf{D_k} = p_k \mathbf{a}\boldsymbol{\alpha}$.

- MMPP (Markov modulated Poisson process):
  $\mathbf{D_0} = \mathbf{Q} - \text{diag}<\boldsymbol{\lambda}>$, $\mathbf{D_1} = \text{diag}<\boldsymbol{\lambda}>$.

- IPP (Interrupted Poisson process):

$$\mathbf{D_0} = \begin{array}{|c|c|}\hline -\alpha-\lambda & \alpha \\ \hline 0 & -\beta \\ \hline \end{array}\,, \qquad \mathbf{D_1} = \begin{array}{|c|c|}\hline \lambda & 0 \\ \hline 0 & 0 \\ \hline \end{array}\,.$$

- batch MMPP :
  $\mathbf{D_0} = \mathbf{Q} - \text{diag}<\boldsymbol{\lambda}>$, $\mathbf{D_k} = p_k\ \text{diag}<\boldsymbol{\lambda}>$.

Batch Markov arrival process

Examples:

- filtered MAP (arrivals discarded with probability p): $\mathbf{D_0} = \hat{\mathbf{D}}_0 + p\hat{\mathbf{D}}_1$, $\mathbf{D_1} = (1-p)\hat{\mathbf{D}}_1$.

- cyclicly filtered MAP (every second arrivals are discarded with probability p):

$$\mathbf{D_0} = \begin{array}{|c|c|}\hline \hat{\mathbf{D}}_0 & 0 \\ \hline p\hat{\mathbf{D}}_1 & \hat{\mathbf{D}}_0 \\ \hline \end{array}, \qquad \mathbf{D_1} = \begin{array}{|c|c|}\hline 0 & \hat{\mathbf{D}}_1 \\ \hline (1-p)\hat{\mathbf{D}}_1 & 0 \\ \hline \end{array}.$$

- superposition of BMAPs: $\mathbf{D_k} = \hat{\mathbf{D}}_k \bigoplus \tilde{\mathbf{D}}_k$,

Kronecker product: $\mathbf{A} \bigotimes \mathbf{B} = \begin{array}{|ccc|}\hline A_{11}\mathbf{B} & \ldots & A_{1n}\mathbf{B} \\ \vdots & & \vdots \\ A_{n1}\mathbf{B} & \ldots & A_{nn}\mathbf{B} \\ \hline \end{array}$

Kronecker sum: $\mathbf{A} \bigoplus \mathbf{B} = \mathbf{A} \bigotimes \mathbf{I_B} + \mathbf{I_A} \bigotimes \mathbf{B}$

Batch Markov arrival process

- Departure process of an M/M/1/2 queue:

$$\mathbf{D_0} = \begin{array}{|c|c|c|}\hline -\lambda & \lambda & \\ \hline & -\lambda-\mu & \lambda \\ \hline & & -\mu \\ \hline \end{array} \qquad \mathbf{D_1} = \begin{array}{|c|c|c|}\hline & & \\ \hline \mu & & \\ \hline & \mu & \\ \hline \end{array}$$

- Departure process of an MAP/M/1/1 queue:

$$\mathbf{D_0} = \begin{array}{|c|c|}\hline \hat{\mathbf{D}}_\mathbf{0} & \hat{\mathbf{D}}_\mathbf{1} \\ \hline 0 & \hat{\mathbf{D}}_\mathbf{0} + \hat{\mathbf{D}}_\mathbf{1} - \mu\mathbf{I} \\ \hline \end{array}, \qquad \mathbf{D_1} = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline \mu\mathbf{I} & 0 \\ \hline \end{array}.$$

- Correlated inter-arrivals ($\lambda_1 \neq \lambda_2$):

$$\mathbf{D_0} = \begin{array}{|c|c|}\hline -\lambda_1 & 0 \\ \hline 0 & -\lambda 2 \\ \hline \end{array} \qquad \mathbf{D_1} = \begin{array}{|c|c|}\hline p\lambda_1 & (1-p)\lambda_1 \\ \hline (1-p)\lambda_2 & p\lambda_2 \\ \hline \end{array}$$

$p \sim 1 \rightarrow$ positive correlated consecutive inter-arrivals

$p \sim 0 \rightarrow$ negative correlated consecutive inter-arrivals

Batch Markov arrival process

Regular BMAPs:

- phase-process ($\mathbf{D}$) is irreducible,
- mean inter-arrival time is positive and finite – $\mathbf{D_0}$ non-singular,
- mean arrival rate, $\mathbf{d} = \sum_{k=0}^{\infty} k\mathbf{D_k}\mathbb{1}$, is finite.

Properties of regular BMAPs:

- $M(t) = E(N(t))$ mean number of arrivals, $m(t) = \frac{d}{dt}M(t)$ arrival rate,
- $\boldsymbol{\pi}$ stationary phase distribution at arrival,
- $\boldsymbol{\alpha}$ stationary phase distribution ($\boldsymbol{\alpha}\mathbf{D} = \mathbf{0}, \boldsymbol{\alpha}\mathbb{1} = 1$)

$$m(t) = \boldsymbol{\pi} e^{\mathbf{D}t}\mathbf{d}, \quad \bar{\lambda} = \lim_{t\to\infty} m(t) = \boldsymbol{\alpha}\mathbf{d},$$

$$M(t) = \int_{x=0}^{t} m(x)dx = \boldsymbol{\alpha}\mathbf{d}\ t + \boldsymbol{\pi}(e^{\mathbf{D}t} - \mathbf{I})(\mathbf{D} - \mathbb{1}\boldsymbol{\alpha})^{-1}\mathbf{d}$$

Batch Markov arrival process

Distribution of the number of arrivals
(short term behaviour):

$$P_{ij}(n,t) = Pr(N(t) = n, J(t) = j | N(0) = 0, J(0) = i)$$

$$\mathbf{P}(n,t) = \{P_{ij}(n,t)\} \quad \text{(matrix)}$$

The transient behaviour:

$$\frac{d}{dt}\mathbf{P}(n,t) = \sum_{k=0}^{n} \mathbf{P}(n-k,t)\mathbf{D_k}$$

initial conditions:

$\mathbf{P}(0,0) = \mathbf{I}$, and $\mathbf{P}(n,0) = \mathbf{0}$ for $n > 0$.

z transform: $\hat{\mathbf{P}}(z,t) = \sum_{n=0}^{\infty} z^n \mathbf{P}(n,t)$.

$$\frac{d}{dt}\hat{\mathbf{P}}(z,t) = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \mathbf{P}(n-k,t)\mathbf{D_k}$$
$$= \sum_{k=0}^{\infty} z^k \sum_{n=k}^{\infty} z^{n-k}\mathbf{P}(n-k,t)\mathbf{D_k} = \hat{\mathbf{P}}(z,t)\mathbf{D(z)}$$

Solution: $\hat{\mathbf{P}}(z,t) = e^{\mathbf{D(z)}t}$, since $\hat{\mathbf{P}}(z,0) = \mathbf{I}$

Batch Markov arrival process

Distribution of the number of arrivals
(regenerative approach):

$$\mathbf{P}(0,t) = e^{\mathbf{D_0}t},$$

$$\mathbf{P}(n,t) = \int_{\tau=0}^{t} e^{\mathbf{D_0}\tau} \sum_{k=1}^{n} \mathbf{D_k}\ \mathbf{P}(n-k,t-\tau)d\tau,\ \ n \geq 1$$

Laplace transform: $\mathbf{P^*(n,s)} = \int_{\mathbf{t=0}}^{\infty} \mathbf{e^{-st}P(n,t)dt}.$

$$\mathbf{P}^*(0,s) = (s\mathbf{I} - \mathbf{D_0})^{-1},$$

$$\mathbf{P}^*(n,s) = (s\mathbf{I} - \mathbf{D_0})^{-1} \sum_{k=1}^{n} \mathbf{D_k}\ \mathbf{P}^*(n-k,s),\ \ n \geq 1$$

z transform: $\widehat{\mathbf{P}}^*(z,s) = \sum_{n=0}^{\infty} z^n \mathbf{P}^*(n,s).$

$$\widehat{\mathbf{P}}^*(z,s) = (s\mathbf{I} - \mathbf{D_0})^{-1}(\mathbf{I} + (\mathbf{D(z)} - \mathbf{D_0})\widehat{\mathbf{P}}^*(z,s)),$$

$$\widehat{\mathbf{P}}^*(z,s) = (s\mathbf{I} - \mathbf{D(z)})^{-1},$$

Inverse Laplace transform:

$$\widehat{\mathbf{P}}(z,t) = e^{\mathbf{D(z)}t}.$$

Quasi birth-death process

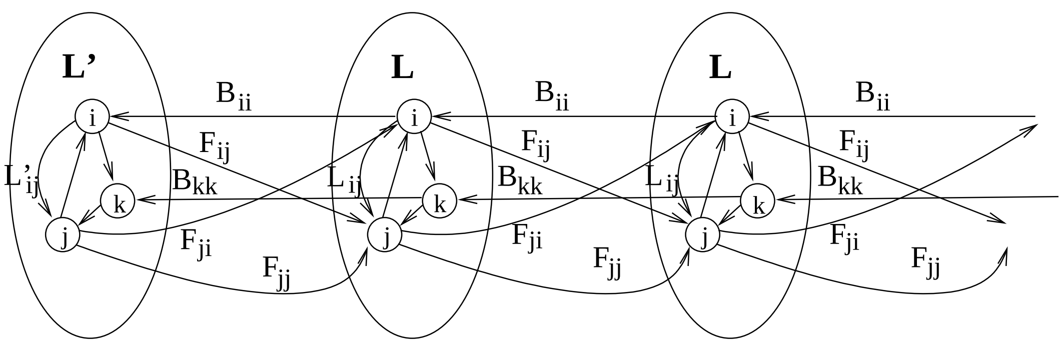
Continuous time QBD:

$\{N(t), J(t)\}$ is a CTMC, where

- $N(t)$ is the "level" process (e.g., number of customers in a queue),

- $J(t)$ is the "phase" process (e.g., state of the environment).

$\{N(t), J(t)\}$ is a Quasi birth-death process if transitions are restricted to one level up or down or inside the same level.



Level 0 is irregular (e.g., no departure).

Quasi birth-death process

Applied notation:

- $\mathbf{F}$ – (forward) transitions one level up (e.g., arrival)
- $\mathbf{L}$ – (local) transitions in the same level
- $\mathbf{B}$ – (backward) transitions one level down (e.g., departure)
- $\mathbf{L}'$ – irregular block at level 0.

(In the L-R book: $\mathbf{F} = \mathbf{A_0}$, $\mathbf{L} = \mathbf{A_1}$, $\mathbf{B} = \mathbf{A_2}$.)

Structure of the generator matrix:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|} \hline \mathbf{L}' & \mathbf{F} & & & \\ \hline \mathbf{B} & \mathbf{L} & \mathbf{F} & & \\ \hline & \mathbf{B} & \mathbf{L} & \mathbf{F} & \\ \hline & & \mathbf{B} & \mathbf{L} & \mathbf{F} \\ \hline & & & \ddots & \ddots \\ \hline \end{array}$$

On the block level it has a birth-death structure.

$\longrightarrow$ "quasi" birth-death process.

Quasi birth-death process

Example: PH/M/1 queue

- arrival process: PH renewal process with representation $\tau, \mathbf{T}$, $(t = -\mathbf{T}\mathbb{1})$
- service time: exponentially distributed with parameter μ.

Structure of the transition probability matrix:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|}
\hline
\mathbf{T} & t\tau & & & \\
\hline
\mu\mathbf{I} & \mathbf{T}-\mu\mathbf{I} & t\tau & & \\
\hline
 & \mu\mathbf{I} & \mathbf{T}-\mu\mathbf{I} & t\tau & \\
\hline
 & & \mu\mathbf{I} & \mathbf{T}-\mu\mathbf{I} & t\tau \\
\hline
 & & & \ddots & \ddots \\
\hline
\end{array}$$

That is $\mathbf{F} = t\tau$, $\mathbf{L} = \mathbf{T} - \mu\mathbf{I}$, $\mathbf{B} = \mu\mathbf{I}$ and $\mathbf{L}' = \mathbf{T}$.

Quasi birth-death process

Example: MAP/PH/1/K queue

- arrival process: MAP $\mathbf{D_0}, \mathbf{D_1}$,
- service time: PH($\tau, \mathbf{T}$), ($t = -\mathbf{T}\mathbb{1}$).

Structure of the transition probability matrix:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|}\hline \mathbf{L}' & \mathbf{F}' & & & \\ \hline \mathbf{B}' & \mathbf{L} & \ddots & & \\ \hline & \mathbf{B} & \ddots & \mathbf{F} & \\ \hline & & \ddots & \mathbf{L} & \mathbf{F} \\ \hline & & & \mathbf{B} & \mathbf{L}^{''} \\ \hline \end{array}$$

Where
$\mathbf{F} = \mathbf{D_1} \bigotimes \mathbf{I}$, $\mathbf{L} = \mathbf{D_0} \bigoplus \mathbf{T}$, $\mathbf{B} = \mathbf{I} \bigotimes t\tau$,
$\mathbf{F}' = \mathbf{D_1} \bigotimes \tau$, $\mathbf{L}' = \mathbf{D_0}$, $\mathbf{B}' = \mathbf{I} \bigotimes \mathbf{T}$ and
$\mathbf{L}^{''} = (\mathbf{D_0} + \mathbf{D_1}) \bigoplus \mathbf{T}$.

Condition of stability

Phase process in the regular part ($n > 1$) is a CTMC with generator matrix:

$$\mathbf{A} = \mathbf{F} + \mathbf{L} + \mathbf{B}$$

Assuming $\mathbf{A}$ is irreducible, the stationary solution of $\mathbf{A}$ is:

$$\boldsymbol{\alpha}\mathbf{A} = \mathbf{0}, \boldsymbol{\alpha}\mathbb{1} = 1$$

The stationary drift of the level process is:

$$d = \boldsymbol{\alpha}\mathbf{F}\mathbb{1} - \boldsymbol{\alpha}\mathbf{B}\mathbb{1}$$

Condition of stability:

$$d = \boldsymbol{\alpha}\mathbf{F}\mathbb{1} - \boldsymbol{\alpha}\mathbf{B}\mathbb{1} < 0$$

Matrix geometric distribution

Stationary solution: $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$, $\boldsymbol{\pi}\mathbb{1} = 1$.

Partitioning $\boldsymbol{\pi}$: $\boldsymbol{\pi} = \{\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots\}$

Decomposed stationary equations:

$$\boldsymbol{\pi}_0\mathbf{L}' + \boldsymbol{\pi}_1\mathbf{B} = \mathbf{0}$$

$$\boldsymbol{\pi}_{n-1}\mathbf{F} + \boldsymbol{\pi}_n\mathbf{L} + \boldsymbol{\pi}_{n+1}\mathbf{B} = \mathbf{0} \qquad \forall n \geq 1$$

$$\sum_{n=0}^{\infty} \boldsymbol{\pi}_n\mathbb{1} = 1$$

Conjecture: $\boldsymbol{\pi}_n = \boldsymbol{\pi}_{n-1}\mathbf{R} \qquad \rightarrow \qquad \boldsymbol{\pi}_n = \boldsymbol{\pi}_0\mathbf{R}^n$

This conjecture gives:

$$\boldsymbol{\pi}_0\mathbf{L}' + \boldsymbol{\pi}_0\mathbf{R}\mathbf{B} = \mathbf{0}$$

$$\boldsymbol{\pi}_0\mathbf{R}^{n-1}\mathbf{F} + \boldsymbol{\pi}_0\mathbf{R}^n\mathbf{L} + \boldsymbol{\pi}_0\mathbf{R}^{n+1}\mathbf{B} = \mathbf{0} \qquad \forall n \geq 1$$

$$\sum_{n=0}^{\infty} \boldsymbol{\pi}_0\mathbf{R}^n\mathbb{1} = \boldsymbol{\pi}_0(\mathbf{I} - \mathbf{R})^{-1}\mathbb{1} = 1$$

Matrix geometric distribution

The solution is defined by vector π_0 and matrix $\mathbf{R}$:

Matrix $\mathbf{R}$ is the solution of the matrix equation:

$$\mathbf{F} + \mathbf{R}\mathbf{L} + \mathbf{R}^2\mathbf{B} = \mathbf{0}$$

Vector π_0 is the solution of linear system:

$$\boldsymbol{\pi}_0(\mathbf{L}' + \mathbf{R}\mathbf{B}) = \mathbf{0}$$

$$\boldsymbol{\pi}_0(\mathbf{I} - \mathbf{R})^{-1}\mathbb{1} = 1$$

Note that $\mathbf{L}' + \mathbf{R}\mathbf{B}$ $(= \mathbf{L}' + \mathbf{F}\mathbf{G})$ is the generator matrix of the restricted process on level 0.

Matrix geometric distribution

Properties of $\mathbf{R}$:

- the matrix equation has more than one solutions.
- if the QBD is stable there is a solution $\mathbf{R}$ whose eigenvalues $(\lambda_i(\mathbf{R}))$ are $|\lambda_i(\mathbf{R})| < 1$ and this is the relevant $\mathbf{R}$ matrix.
- (if the QBD is not stable there is a solution $\mathbf{R}$ whose eigenvalues $(\lambda_i(\mathbf{R}))$ are $|\lambda_i(\mathbf{R})| \leq 1$ and this is the relevant $\mathbf{R}$ matrix.)

Stochastic interpretation:

$\mathbf{R}_{ij}$ is the ratio of the mean time spent in (n, j) and the mean time spent in $(n-1, i)$ before the first return to level $n-1$ starting from $(n-1, i)$.

In a homogeneous QBD $\mathbf{R}_{ij}$ is independent of n.

Analysis of the level process

Example: busy period of the M/M/1 queue

The busy period of the M/M/1 queue starts when a customer arrives to an idle system, and it ends when the system becomes idle again, i.e., "the level process moves from 1 to 0"

Let T be the time of the busy period and $g(s) = E(e^{-sT})$ its Laplace transform.

$$g(s) = \frac{\mu}{\lambda+\mu}\,\frac{\lambda+\mu}{\lambda+\mu+s} + \frac{\lambda}{\lambda+\mu}\left(\frac{\lambda+\mu}{\lambda+\mu+s}g^2(s)\right)$$

At the beginning of the busy period the process stays exp. $(\lambda+\mu)$ time at level 1. After that it moves to level 0 with probability $\dfrac{\mu}{\lambda+\mu}$ and to level 2 with probability $\dfrac{\lambda}{\lambda+\mu}$.

From level 2, it returns to level 0 in two steps: from level 2 to 1 and from level 1 to 0.

Due to the homogeneous structure of the chain these two times are i.i.d.

Analysis of the level process

γ_n denotes the time of the first visit to level n:

$\gamma_n = \min(t|t > 0, N(t) = n)$

First visit from level n to $n-1$:

$G_{ij}(t) = Pr(J(\gamma_{n-1}) = j, \gamma_{n-1} < t | N(0) = n, J(0) = i)$

$$g_{ij}(t) = \frac{d}{dt} G_{ij}(t), \quad G^{\sim}_{ij}(s) = \int_{t=0}^{\infty} e^{-st} g_{ij}(t) dt.$$

Transition from level n to $n-1$:

- direct step down,
- transition inside level n,
- transition to level $n+1$:

$$G^{\sim}_{ij}(s) = \frac{-\mathbf{L}_{ii}}{s - \mathbf{L}_{ii}} \left(\frac{\mathbf{B}_{ij}}{-\mathbf{L}_{ii}} + \sum_{k \in S, k \neq i} \frac{\mathbf{L}_{ik}}{-\mathbf{L}_{ii}} G^{\sim}_{kj}(s) \right.$$
$$\left. + \sum_{k \in S} \frac{\mathbf{F}_{ik}}{-\mathbf{L}_{ii}} \sum_{\ell \in S} G^{\sim}_{k\ell}(s) G^{\sim}_{\ell j}(s) \right),$$

that is

$$\mathbf{0} = \mathbf{B} + (\mathbf{L} - s\mathbf{I}) \mathbf{G}^{\sim}(s) + \mathbf{F}\mathbf{G}^{\sim 2}(s).$$

Analysis of the level process

First state visited in level $n-1$ starting from level n:

$$G_{ij} = Pr(J(\gamma_{n-1}) = j | N(0) = n, J(0) = i)$$

$$G_{ij} = \lim_{t \to \infty} G_{ij}(t) \quad \Rightarrow \quad G_{ij} = \lim_{s \to 0} G_{ij}^{\sim}(s) \quad \Rightarrow$$

$$\mathbf{0} = \mathbf{B} + \mathbf{LG} + \mathbf{FG}^2$$

Restricted process

The state space of the irreducible CTMC with generator $\mathbf{Q}$ is divided into disjoint subset $\mathcal{U}$ and $\mathcal{D}$.

The decomposed generator matrix is

$$\mathbf{Q} = \begin{array}{|c|c|} \hline \mathbf{Q_1} & \mathbf{Q_2} \\ \hline \mathbf{Q_3} & \mathbf{Q_4} \\ \hline \end{array} \, .$$

Restricted process:

We study the process during its visits to $\mathcal{U}$. I.e. the clock is stopped when the CTMC visits $\mathcal{D}$ and is resumed when it returns to $\mathcal{U}$.

The obtained restricted process is a CTMC with generator

$$\mathbf{Q_U} = \mathbf{Q_1} + \mathbf{Q_2} \mathbf{P}_{\mathcal{D}\to\mathcal{U}}$$

where for $i \in \mathcal{D}$ and $j \in \mathcal{U}$

$$[\mathbf{P}_{\mathcal{D}\to\mathcal{U}}]_{ij} = Pr(X(\gamma_{\mathcal{U}}) = j \mid X(0) = i)$$

and $\gamma_{\mathcal{U}}$ is the fist time when the process visits $\mathcal{U}$.

From $\mathbf{P}_{\mathcal{D}\to\mathcal{U}} = (-\mathbf{Q_4})^{-1}\mathbf{Q_3}$ we have

$$\mathbf{Q_U} = \mathbf{Q_1} + \mathbf{Q_2}(-\mathbf{Q_4})^{-1}\mathbf{Q_3}$$

The restricted process is also referred to as stochastic complement.

Analysis of the level process

Time spent at level n before visiting level $n-1$:

We consider $\{N(t), J(t)\}$ which starts at level n ($N(0) = n$) and terminates when $N(t) = n-1$.

Restricted process on level n:
time (clock) increases as long as $N(t) = n$ and it stops when $N(t) > n$.

The restricted process is a CTMC with generator

$$\mathbf{U} = \mathbf{L} + \mathbf{F}\mathbf{G}.$$

The mean time spent in level n is characterized by $(-\mathbf{U})^{-1}$, hence

$$\mathbf{U} = \mathbf{L} + \mathbf{F}(-\mathbf{U})^{-1}\mathbf{B}$$

where $\mathbf{L}$ stands for the transitions inside level n and $\mathbf{F}(-\mathbf{U})^{-1}\mathbf{B}$ describes the effect of transitions to level $n+1$ and the first return to level n.

Analysis of the level process

$E(\mathcal{T}_{ij})$ – mean time spent in state $\{n+1, j\}$ before the first jump to level n starting from $\{n, i\}$.

$\mathcal{T}_{ij}$ is 0, if a transition to level $n-1$ or to level n takes place, hence

$$E(\mathcal{T}_{ij}) = \sum_k \frac{\mathbf{F}_{ik}}{-\mathbf{L}_{ii}} \ (-\mathbf{U})^{-1}_{kj}$$

Mean time spent in state $\{n, i\}$ is $\dfrac{1}{-\mathbf{L}_{ii}}$.

The ratio of time spent in $\{n+1, j\}$ and in $\{n, i\}$ before a jump to level n is

$$\mathbf{R}_{ij} = \frac{\displaystyle\sum_k \frac{\mathbf{F}_{ik}}{-\mathbf{L}_{ii}} \ (-\mathbf{U})^{-1}_{kj}}{\dfrac{1}{-\mathbf{L}_{ii}}} = \sum_k \mathbf{F}_{ik} \ (-\mathbf{U})^{-1}_{kj}.$$

That is

$$\mathbf{R} = \mathbf{F} \ (-\mathbf{U})^{-1}.$$

Matrix geometric distribution

Summary:
Matrices describing a QBD:

Matrix $\mathbf{R}$:
"Ratio of time spent at level n and level $n+1$ before the first return to level n"

$$\mathbf{F} + \mathbf{R}\mathbf{L} + \mathbf{R}^2\mathbf{B} = \mathbf{0}$$

Matrix $\mathbf{G}$:
"The first state visited at level $n-1$ starting from n"

$$\mathbf{G}_{ij} = Pr(J(\gamma_{n-1}) = j | N_0 = n, J_0 = i)$$

$$\mathbf{F}\mathbf{G}^2 + \mathbf{L}\mathbf{G} + \mathbf{B} = \mathbf{0}$$

Matrix $\mathbf{U}$:
"Generator of the QBD process restricted to level n before returning to $n-1$"

$\rightarrow$ $(-\mathbf{U})^{-1}$ "mean time spent at level n before visiting level $n-1$"

$$\mathbf{U} = \mathbf{L} + \mathbf{F}(-\mathbf{U})^{-1}\mathbf{B}$$

Matrix geometric distribution

Relation of matrices describing a QBD:

$$\begin{aligned}\mathbf{U} &= \mathbf{L} + \mathbf{F}\mathbf{G} \\ &= \mathbf{L} + \mathbf{R}\mathbf{B}\end{aligned}$$

$$\begin{aligned}\mathbf{G} &= (-\mathbf{U})^{-1}\mathbf{B} \\ &= (-\mathbf{L} - \mathbf{R}\mathbf{B})^{-1}\mathbf{B}\end{aligned}$$

$$\begin{aligned}\mathbf{R} &= \mathbf{F}(-\mathbf{U})^{-1} \\ &= \mathbf{F}(-\mathbf{L} - \mathbf{F}\mathbf{G})^{-1}\end{aligned}$$

Matrix geometric distribution

Properties of $\mathbf{R}$, $\mathbf{U}$ and $\mathbf{G}$ in a *stable* QBD:

- $\mathbf{R}$: "ratio of mean time spent in level $n+1$ and n" ($0 \leq \mathbf{R}_{ij}$).

  The eigenvalues ($\lambda_i(\mathbf{R})$) are $|\lambda_i(\mathbf{R})| < 1$.

- $\mathbf{U}$: is an incomplete generator matrix (for $i \neq j$, $0 \leq \mathbf{U}_{ij}$, $0 \geq \mathbf{U}_{ii}$, $\mathbf{U}\mathbb{1} \leq \mathbf{0}$).

- $\mathbf{G}$: is a stochastic matrix ($0 \leq \mathbf{G}_{ij} \leq 1$, $\mathbf{G}\mathbb{1} = \mathbb{1}$).

The properties of $\mathbf{G}$ are easier to check and

$$\mathbf{R} = \mathbf{F}\left(-\mathbf{L} - \mathbf{F}\mathbf{G}\right)^{-1}$$

Transient measure for $\mathbf{U}$

Sojourn probability in level n before moving to level $n-1$:

$V_{ij}(t) = Pr(N(t) = n, J(t) = j, \gamma_{n-1} > t | N(0) = n, J(0) = i)$

$$V_{ij}(t|H = h) =$$
$$\begin{cases} \delta_{ij} & h > t \\ \displaystyle\sum_{k \neq i} \frac{L_{ik}}{-L_{ii}} V_{kj}(t-h) + & \\ \displaystyle + \sum_k \sum_\ell \int_{\tau=0}^{t-h} \frac{F_{ik}}{-L_{ii}} g_{k\ell}(\tau) V_{\ell j}(t-h-\tau) d\tau & h < t \end{cases}$$

where $g_{ij}(t) = \frac{d}{dt} G_{ij}(t)$. Applying the law of total probability, $V_{ij}(t) = \int_{h=0}^{\infty} -L_{ii} e^{L_{ii}h} V_{ij}(t|H = h) dh$, gives

$$\begin{aligned} V_{ij}(t) = & \int_{h=t}^{\infty} -L_{ii} e^{L_{ii}h} \delta_{ij} dh \\ & + \int_{h=0}^{t} -L_{ii} e^{L_{ii}h} \Bigg(\sum_{k \neq i} \frac{L_{ik}}{-L_{ii}} V_{kj}(t-h) + \\ & \qquad + \sum_k \sum_\ell \int_{\tau=0}^{t-h} \frac{F_{ik}}{-L_{ii}} g_{k\ell}(\tau) V_{\ell j}(t-h-\tau) d\tau \Bigg) dh. \end{aligned}$$

Relation of $\mathbf{V}^\star(\mathbf{s})$, $\mathbf{U}$ and $\mathbf{G}^\sim(\mathbf{s})$

Its Laplace transform, $V_{ij}^\star(s) = \int_{t=0}^\infty e^{-st} V_{ij}(t)dt$, gives

$$\mathbf{V}^\star(s) = \left(s\mathbf{I} - \mathbf{L} - \mathbf{F}\,\underbrace{\mathbf{G}^\sim(s)}_{\mathbf{g}^\star(s)} \right)^{-1}.$$

The mean time spent in (n, j) is

$$\int_{t=0}^\infty V_{ij}(t)dt = \lim_{s\to 0} V_{ij}^\star(s) = (-\mathbf{L} - \mathbf{F}\mathbf{G})_{ij}^{-1} = (-\mathbf{U})_{ij}^{-1}.$$

By definition

$$g_{ij}(t) = \sum_k V_{ik}(t) B_{kj},$$

and consequently

$$\mathbf{g}^\star(s) = \mathbf{G}^\sim(s) = \mathbf{V}^\star(s)\mathbf{B}.$$

Transient measure for $\mathbf{R}$

Sojourn probability in level $n+1$ before returning to level n:

$$R_{ij}(t) = \lim_{\Delta \to 0} \frac{1}{\Delta} Pr\Big(N(t) = n+1, J(t) = j, \gamma_n > t,$$
$$\text{transition in } (-\Delta, 0)$$
$$\mid N(-\Delta) = n, J(-\Delta) = i \Big) =$$
$$= \sum_k F_{ik} V_{kj}(t).$$

From which $R^{\star}_{ij}(s) = \int_{t=0}^{\infty} e^{-st} R_{ij}(t) dt$ is

$$\mathbf{R}^{\star}(s) = \mathbf{F}\mathbf{V}^{\star}(s)$$

and the mean time spent in $(n+1, j)$ is

$$\int_{t=0}^{\infty} R_{ij}(t) dt = \lim_{s \to 0} R^{\star}_{ij}(s) = \mathbf{R}_{ij}.$$

Summary of transient measures

From

$$\mathbf{F}\mathbf{V}^\star(s)\mathbf{B} = \mathbf{R}^\star(s)\mathbf{B} = \mathbf{F}\mathbf{G}^\sim(s),$$

we have

$$\mathbf{V}^\star(s) = \left(s\mathbf{I} - \mathbf{L} - \mathbf{F}\mathbf{V}^\star(s)\mathbf{B}\right)^{-1},$$

$$s\mathbf{R}^\star(s) = \mathbf{F} + \mathbf{R}^\star(s)\mathbf{L} + \mathbf{R}^{\star 2}(s)\mathbf{B},$$

and the quadratic equation for $\mathbf{G}^\sim(s)$ on page 123.

Algorithms to compute $\mathbf{R}/\mathbf{G}$

- Linear
- Quadratic
  - Logarithmic reduction
  - Newton's iteration
  - Cyclic reduction

Linear algorithms

Linear progression algorithm to calculate $\mathbf{G}$:

$$
\begin{array}{l}
\mathbf{G} := \mathbf{0}; \\
\textbf{REPEAT} \\
\quad \mathbf{G} := (-\mathbf{L} - \mathbf{F}\mathbf{G})^{-1}\,\mathbf{B}; \\
\textbf{UNTIL} ||\mathbb{1} - \mathbf{G}\mathbb{1}|| \leq \epsilon
\end{array}
$$

Linear boundary algorithm to calculate $\mathbf{G}$:

$$
\begin{array}{l}
\mathbf{G} := \mathbf{I}; \\
\textbf{REPEAT} \\
\quad \mathbf{G}_{old} := \mathbf{G}; \\
\quad \mathbf{G} := (-\mathbf{L} - \mathbf{F}\mathbf{G})^{-1}\,\mathbf{B}; \\
\textbf{UNTIL} ||\mathbf{G} - \mathbf{G}_{old}|| \leq \epsilon
\end{array}
$$

Linear algorithm to calculate $\mathbf{R}$:

$$
\begin{array}{l}
\mathbf{R} := \mathbf{0}; \\
\textbf{REPEAT} \\
\quad \mathbf{R}_{old} := \mathbf{R}; \\
\quad \mathbf{R} := \mathbf{F}\,(-\mathbf{L} - \mathbf{R}\mathbf{B})^{-1}\,; \\
\textbf{UNTIL} ||\mathbf{R} - \mathbf{R}_{old}|| \leq \epsilon
\end{array}
$$

Linear algorithms

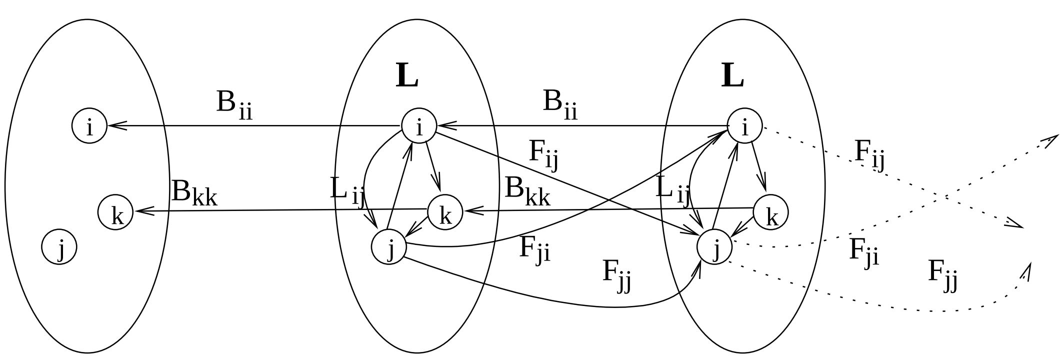
Stochastic interpretation of the linear progression algorithm after i iterations:

$$\begin{aligned}
&\mathbf{G_0} = \mathbf{0},\\
&\mathbf{G_1} = (-\mathbf{L})^{-1}\mathbf{B},\\
&\mathbf{G_2} = (-\mathbf{L} - \mathbf{F}(-\mathbf{L})^{-1}\mathbf{B})^{-1}\mathbf{B},\\
&\mathbf{G_3} = \ldots
\end{aligned}$$

where

$$\mathbf{Q_1} = \begin{array}{|c|c|} \hline & \\ \hline \mathbf{B} & \mathbf{L} \\ \hline \end{array}\,, \quad \mathbf{Q_2} = \begin{array}{|c|c|c|} \hline & & \\ \hline \mathbf{B} & \mathbf{L} & \mathbf{F} \\ \hline & \mathbf{B} & \mathbf{L} \\ \hline \end{array}\,, \quad \mathbf{Q_3} = \ldots$$

$$[\mathbf{G_n}]_{ij} = Pr(J(\gamma_0) = j, \gamma_0 < \gamma_{n+1} | N_0 = 1, J_0 = i)$$



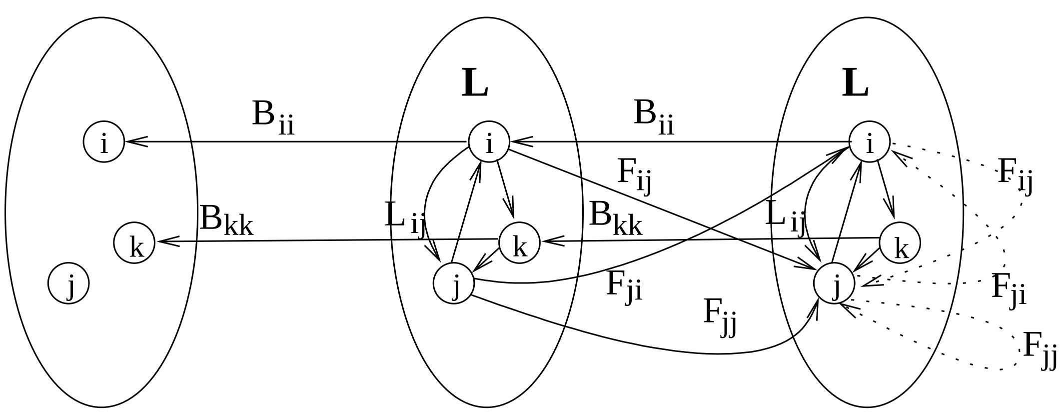
Linear algorithms

Stochastic interpretation of the linear boundary algorithm after i iterations:

$$\begin{aligned}
\mathbf{G_0} &= \mathbf{I}, \\
\mathbf{G_1} &= (-\mathbf{L} - \mathbf{F})^{-1}\mathbf{B}, \\
\mathbf{G_2} &= (-\mathbf{L} - \mathbf{F}(-\mathbf{L} - \mathbf{F})^{-1}\mathbf{B})^{-1}\mathbf{B}, \\
\mathbf{G_3} &= \ldots
\end{aligned}$$

where

$$\mathbf{Q_1} = \begin{array}{|c|c|} \hline & \\ \hline \mathbf{B} & \mathbf{L+F} \\ \hline \end{array}, \quad \mathbf{Q_2} = \begin{array}{|c|c|c|} \hline & & \\ \hline \mathbf{B} & \mathbf{L} & \mathbf{F} \\ \hline & \mathbf{B} & \mathbf{L+F} \\ \hline \end{array}, \quad \mathbf{Q_3} = \ldots$$



Discrete time Quasi birth-death process (DTQBD)

$\{N(t), J(t)\}$ is a DTMC, where

- $N(t)$ is the "level" process (e.g., number of customers in a queue),
- $J(t)$ is the "phase" process (e.g., state of the environment).

Structure of the transition probability matrix:

$$\mathbf{P} = \begin{array}{|c|c|c|c|c|} \hline \mathbf{L}' & \mathbf{F} & & & \\ \hline \mathbf{B} & \mathbf{L} & \mathbf{F} & & \\ \hline & \mathbf{B} & \mathbf{L} & \mathbf{F} & \\ \hline & & \mathbf{B} & \mathbf{L} & \mathbf{F} \\ \hline & & & \ddots & \ddots \\ \hline \end{array}$$

$\mathbf{B} + \mathbf{L} + \mathbf{F}$ is a stochastic matrix,

$\mathbf{L}' + \mathbf{F}$ as well.

Condition of stability (DTQBD)

Phase process in the regular part ($n > 1$) is a DTMC with generator matrix:

$$\mathbf{P_J} = \mathbf{F} + \mathbf{L} + \mathbf{B}$$

Assuming $\mathbf{P_J}$ is irreducible, the stationary solution of $\mathbf{P_J}$ is the solution of:

$$\boldsymbol{\alpha}\mathbf{P_J} = \boldsymbol{\alpha}, \boldsymbol{\alpha}\mathbb{1} = 1$$

The stationary drift of the level process is:

$$d = \boldsymbol{\alpha}\mathbf{F}\mathbb{1} - \boldsymbol{\alpha}\mathbf{B}\mathbb{1}$$

Condition of stability:

$$d = \boldsymbol{\alpha}\mathbf{F}\mathbb{1} - \boldsymbol{\alpha}\mathbf{B}\mathbb{1} < 0$$

Matrix geometric distribution (DTQBD)

Stationary solution: $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$, $\boldsymbol{\pi}\mathbb{1} = 1$.

Partitioning $\boldsymbol{\pi}$: $\boldsymbol{\pi} = \{\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots\}$

Decomposed stationary equations:

$$\boldsymbol{\pi}_0\mathbf{L}' + \boldsymbol{\pi}_1\mathbf{B} = \boldsymbol{\pi}_0$$

$$\boldsymbol{\pi}_{n-1}\mathbf{F} + \boldsymbol{\pi}_n\mathbf{L} + \boldsymbol{\pi}_{n+1}\mathbf{B} = \boldsymbol{\pi}_n \qquad \forall n \geq 1$$

$$\sum_{n=0}^{\infty} \boldsymbol{\pi}_n\mathbb{1} = 1$$

The solution is $\boldsymbol{\pi}_n = \boldsymbol{\pi}_0\mathbf{R}^n$, where

Matrix $\mathbf{R}$ is the solution of the matrix equation

$$\mathbf{F} + \mathbf{R}\mathbf{L} + \mathbf{R}^2\mathbf{B} = \mathbf{R}$$

that is

$$\mathbf{F} + \mathbf{R}(\mathbf{L} - \mathbf{I}) + \mathbf{R}^2\mathbf{B} = \mathbf{0}$$

Vector π_0 is the solution of linear system:

$$\boldsymbol{\pi}_0(\mathbf{L}' + \mathbf{R}\mathbf{B}) = \boldsymbol{\pi}_0$$

$$\boldsymbol{\pi}_0(\mathbf{I} - \mathbf{R})^{-1}\mathbb{1} = 1$$

($\mathbf{L}' + \mathbf{R}\mathbf{B} = \mathbf{L}' + \mathbf{F}\mathbf{G}$ is the transition probability matrix of the restricted process on level 0.)

Analysis of the level process (DTQBD)

First state visited in level $n-1$

starting from level n:

$$G_{ij} = Pr(J(\gamma_{n-1}) = j, \gamma_{n-1} < \infty | N(0) = n, J(0) = i)$$

From the stochastic interpretation

$$G_{ij} = B_{ij} + \sum_k L_{ik} G_{kj} + \sum_k \sum_\ell F_{ik} G_{k\ell} G_{\ell j}$$

from which

$$\mathbf{0} = \mathbf{B} + (\mathbf{L} - \mathbf{I})\mathbf{G} + \mathbf{F}\mathbf{G}^2$$

Restricted process (DTQBD)

The state space of the irreducible DTMC with transition probability matrix $\mathbf{P}$ is divided into disjoint subset $\mathcal{U}$ and $\mathcal{D}$.

The decomposed generator matrix is

$$\mathbf{P} = \begin{array}{|c|c|} \hline \mathbf{P_1} & \mathbf{P_2} \\ \hline \mathbf{P_3} & \mathbf{P_4} \\ \hline \end{array} \; .$$

Restricted process:

We study the process during its visits to $\mathcal{U}$. I.e. the clock is stopped when the DTMC visits $\mathcal{D}$ and is resumed when it returns to $\mathcal{U}$.

The obtained restricted process is a DTMC with transition probability matrix

$$\mathbf{P_U} = \mathbf{P_1} + \mathbf{P_2}\mathbf{P}_{\mathcal{D}\to\mathcal{U}}$$

where for $i \in \mathcal{D}$ and $j \in \mathcal{U}$

$$[\mathbf{P}_{\mathcal{D}\to\mathcal{U}}]_{ij} = Pr(X(\gamma_{\mathcal{U}}) = j \mid X(0) = i)$$

and $\gamma_{\mathcal{U}}$ is the fist time when the process visits $\mathcal{U}$.

Restricted process (DTQBD)

The mean time spent in the states of $\mathcal{D}$ during a visit to $\mathcal{D}$ is $\sum_i \mathbf{P_4}^i = (\mathbf{I} - \mathbf{P}_4)^{-1}$.

From $\mathbf{P}_{\mathcal{D}\to\mathcal{U}} = (\mathbf{I} - \mathbf{P}_4)^{-1}\mathbf{P_3}$ we have

$$\mathbf{P_U} = \mathbf{P_1} + \mathbf{P_2}(\mathbf{I} - \mathbf{P}_4)^{-1}\mathbf{P_3}$$

Analysis of the level process (DTQBD)

Time spent at level n before visiting level $n-1$:

We consider $\{N(t), J(t)\}$ which starts at level n ($N(0) = n$) and terminates when $N(t) = n-1$.

Restricted process on level n:
time (clock) increases as long as $N(t) = n$ and it stops when $N(t) > n$.

The restricted process is a DTMC with generator

$$\mathbf{U} = \mathbf{L} + \mathbf{F}\mathbf{G}.$$

The mean time spent in level n is characterized by $(\mathbf{I} - \mathbf{U})^{-1}$, hence

$$\mathbf{U} = \mathbf{L} + \mathbf{F}(\mathbf{I} - \mathbf{U})^{-1}\mathbf{B}$$

where $\mathbf{L}$ stands for the transitions inside level n and $\mathbf{F}(\mathbf{I} - \mathbf{U})^{-1}\mathbf{B}$ describes the effect of transitions to level $n+1$ and the first return to level n.

Characteristic matrixes of DTQBDs (DTQBD)

$$\mathbf{F} + \mathbf{R}\mathbf{L} + \mathbf{R}^2\mathbf{B} = \mathbf{R}$$

$$\mathbf{F}\mathbf{G}^2 + \mathbf{L}\mathbf{G} + \mathbf{B} = \mathbf{G}$$

$$\mathbf{U} = \mathbf{L} + \mathbf{F}(\mathbf{I} - \mathbf{U})^{-1}\mathbf{B}$$

Relation of matrices describing a QBD:

$$\begin{aligned} \mathbf{U} &= \mathbf{L} + \mathbf{F}\mathbf{G} \\ &= \mathbf{L} + \mathbf{R}\mathbf{B} \end{aligned}$$

$$\begin{aligned} \mathbf{G} &= (\mathbf{I} - \mathbf{U})^{-1}\mathbf{B} \\ &= (\mathbf{I} - \mathbf{L} - \mathbf{R}\mathbf{B})^{-1}\mathbf{B} \end{aligned}$$

$$\begin{aligned} \mathbf{R} &= \mathbf{F}(\mathbf{I} - \mathbf{U})^{-1} \\ &= \mathbf{F}(\mathbf{I} - \mathbf{L} - \mathbf{F}\mathbf{G})^{-1} \end{aligned}$$

End of discrete time QBDs!!

Logarithmic reduction algorithms

Logarithmic reduction algorithm to calculate $\mathbf{G}$:

$$
\begin{array}{l}
\mathbf{H} := (-\mathbf{L})^{-1}\,\mathbf{F}; \\
\mathbf{K} := (-\mathbf{L})^{-1}\,\mathbf{B}; \\
\mathbf{G} := \mathbf{K}; \\
\mathbf{T} := \mathbf{H}; \\
\textbf{REPEAT} \\
\quad \mathbf{U} := \mathbf{HK} + \mathbf{KH}; \\
\quad \mathbf{H} := (\mathbf{I} - \mathbf{U})^{-1}\,\mathbf{H}^2; \\
\quad \mathbf{K} := (\mathbf{I} - \mathbf{U})^{-1}\,\mathbf{K}^2; \\
\quad \mathbf{G} := \mathbf{G} + \mathbf{TK}; \\
\quad \mathbf{T} := \mathbf{TH}; \\
\textbf{UNTIL} ||\mathbb{1} - \mathbf{G}\mathbb{1}|| \leq \epsilon
\end{array}
$$

Logarithmic reduction algorithm to calculate $\mathbf{R}$:

$$
\begin{array}{l}
\mathbf{H} := \mathbf{F}\,(-\mathbf{L})^{-1}; \\
\mathbf{K} := \mathbf{B}\,(-\mathbf{L})^{-1}; \\
\mathbf{R} := \mathbf{H}; \\
\mathbf{T} := \mathbf{K}; \\
\textbf{REPEAT} \\
\quad \mathbf{R}_{old} := \mathbf{R}; \\
\quad \mathbf{U} := \mathbf{HK} + \mathbf{KH}; \\
\quad \mathbf{H} := \mathbf{H}^2\,(\mathbf{I} - \mathbf{U})^{-1}; \\
\quad \mathbf{K} := \mathbf{K}^2\,(\mathbf{I} - \mathbf{U})^{-1}; \\
\quad \mathbf{R} := \mathbf{R} + \mathbf{HT}; \\
\quad \mathbf{T} := \mathbf{KT}; \\
\textbf{UNTIL} ||\mathbf{R} - \mathbf{R}_{old}|| \leq \epsilon
\end{array}
$$

Logarithmic reduction algorithms for $\mathbf{G}$

Stochastic interpretation

Embedded discrete time QBD at level changes

$$\begin{aligned}
&\mathbf{B}'(0) = (-\mathbf{L})^{-1}\mathbf{B}, \\
&\mathbf{F}'(0) = (-\mathbf{L})^{-1}\mathbf{F}, \\
&\mathbf{L}'(0) = \mathbf{0},
\end{aligned}$$

and we have $\mathbf{G}(\mathbf{0}) = \mathbf{G}$ from which

$$\mathbf{G}(0) = \mathbf{B}'(\mathbf{0}) + \underbrace{\mathbf{L}'(\mathbf{0})\mathbf{G}(0)}_{\mathbf{0}} + \mathbf{F}'(\mathbf{0})\mathbf{G}(0)^2 \ .$$

Approximation of $\mathbf{G}$

$$\tilde{\mathbf{G}}(0) = \underbrace{\mathbf{B}'(\mathbf{0})}_{\gamma_0 < \gamma_2}$$

Logarithmic reduction algorithms for $\mathbf{G}$

Discrete time QBD at entrance of odd (2k+1) levels:

$$\begin{aligned}
&\mathbf{B}(1) = \mathbf{B}'(0)^2,\\
&\mathbf{F}(1) = \mathbf{F}'(0)^2,\\
&\mathbf{L}(1) = \mathbf{B}'(0)\mathbf{F}'(0) + \mathbf{F}'(0)\mathbf{B}'(0),
\end{aligned}$$

for this process

$$\mathbf{G}(1) \quad = \mathbf{B}(1) + \mathbf{L}(1)\mathbf{G}(1) + \mathbf{F}(1)\mathbf{G}(1)^2$$

and $\mathbf{G}(1) = \mathbf{G}(0)^2$.

Discrete time QBD process at entrance of odd levels, where the level is changes:

$$\begin{aligned}
&\mathbf{B}'(1) = (\mathbf{I} - \mathbf{L}(1))^{-1}\mathbf{B}(1),\\
&\mathbf{F}'(1) = (\mathbf{I} - \mathbf{L}(1))^{-1}\mathbf{F}(1),\\
&\mathbf{L}'(1) = \mathbf{0},
\end{aligned}$$

for this process

$$\mathbf{G}(1) \quad = \mathbf{B}'(1) + \mathbf{F}'(1)\mathbf{G}(1)^2$$

That is

$$\begin{aligned}
\mathbf{G}(0) \quad &= \mathbf{B}'(\mathbf{0}) + \mathbf{F}'(\mathbf{0})\mathbf{G}(1)\\
&= \underbrace{\mathbf{B}'(\mathbf{0})}_{\gamma_0<\gamma_2} + \underbrace{\mathbf{F}'(\mathbf{0})\mathbf{B}'(1)}_{\gamma_2<\gamma_0<\gamma_4} + \underbrace{\mathbf{F}'(\mathbf{0})\mathbf{F}'(1)\mathbf{G}(1)^2}_{\gamma_4<\gamma_0}.
\end{aligned}$$

Approximation of $\mathbf{G}$

$$\tilde{\mathbf{G}}(1) = \underbrace{\tilde{\mathbf{G}}(0) + \mathbf{F}'(\mathbf{0})\mathbf{B}'(1)}_{\gamma_0<\gamma_4}$$

Logarithmic reduction algorithms for $\mathbf{G}$

Process at entrance of $2^n k + 1$ levels:

$$\begin{aligned}
&\mathbf{B}(n) = \mathbf{B}'(n-1)^2,\\
&\mathbf{F}(n) = \mathbf{F}'(n-1)^2,\\
&\mathbf{L}(n) = \mathbf{B}'(n-1)\mathbf{F}'(n-1) + \mathbf{F}'(n-1)\mathbf{B}'(n-1).
\end{aligned}$$

for this process

$$\mathbf{G}(n) \quad = \mathbf{B}(n) + \mathbf{L}(n)\mathbf{G}(n) + \mathbf{F}(n)\mathbf{G}(n)^2$$

where $\mathbf{G}(n) = \mathbf{G}(n-1)^2$.

Discrete time QBD process at entrance of $2^n k+1$ levels, where the level is changes:

$$\begin{aligned}
&\mathbf{B}'(n) = (\mathbf{I} - \mathbf{L}(n))^{-1}\mathbf{B}(n),\\
&\mathbf{F}'(n) = (\mathbf{I} - \mathbf{L}(n))^{-1}\mathbf{F}(n),\\
&\mathbf{L}'(n) = \mathbf{0}.
\end{aligned}$$

for this process

$$\mathbf{G}(n) \quad = \mathbf{B}'(n) + \mathbf{F}'(n)\mathbf{G}(n)^2$$

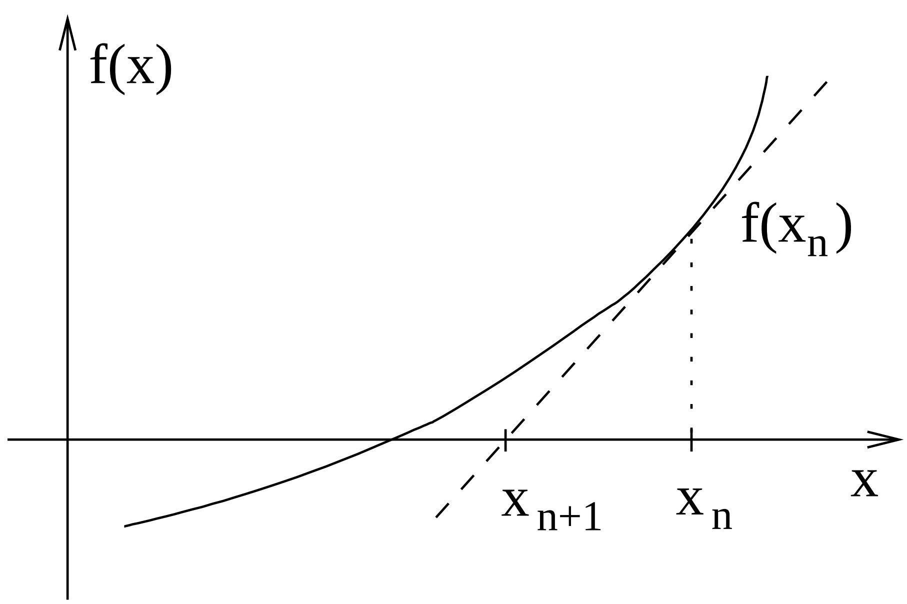
Approximation of $\mathbf{G}$

$$\tilde{\mathbf{G}}(n) \quad = \underbrace{\tilde{\mathbf{G}}(n-1)}_{\gamma_0 < \gamma_{2^n}} + \underbrace{\prod_{i=1}^{n-1} \mathbf{F}'(\mathbf{i})\mathbf{B}'(n)}_{\gamma_{2^n} < \gamma_0 < \gamma_{2^{n+1}}}$$

Newtons' iterations

To solve $f(x) = 0$ start from x_0 and do

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



Newtons' iterations

The same approach applicable for operators. Let $\mathcal{G}$: $\mathbf{X} \to \mathbf{B} + \mathbf{L}\mathbf{X} + \mathbf{F}\mathbf{X}^2$ and find $\mathcal{G}\mathbf{X} = \mathbf{0}$.

The Gateaux derivative of $\mathcal{G}$ at point $\mathbf{X}$ is also an operator. It is defined as

$$\mathcal{G}'(\mathbf{X}) : \mathbf{H} \to \lim_{\tau \to 0} \frac{\mathcal{G}(\mathbf{X} + \tau\mathbf{H}) - \mathcal{G}\mathbf{X}}{\tau}$$

According to this definition

$$\begin{aligned}
&\mathcal{G}'(\mathbf{X}) : \\
&\mathbf{H} \to \lim_{\tau \to 0} \frac{\mathcal{G}(\mathbf{X} + \tau\mathbf{H}) - \mathcal{G}\mathbf{X}}{\tau} \\
&\mathbf{H} \to \lim_{\tau \to 0} \frac{\mathbf{B} + \mathbf{L}(\mathbf{X} + \tau\mathbf{H}) + \mathbf{F}(\mathbf{X} + \tau\mathbf{H})^2 - (\mathbf{B} + \mathbf{L}\mathbf{X} + \mathbf{F}\mathbf{X}^2)}{\tau} \\
&\mathbf{H} \to \lim_{\tau \to 0} \frac{\mathbf{L}\tau\mathbf{H} + \mathbf{F}(\mathbf{X}^2 + \mathbf{X}\tau\mathbf{H} + \tau\mathbf{H}\mathbf{X} + \tau^2\mathbf{H}^2) - \mathbf{F}\mathbf{X}^2}{\tau} \\
&\mathbf{H} \to \mathbf{L}\mathbf{H} + \mathbf{F}(\mathbf{X}\mathbf{H} + \mathbf{H}\mathbf{X})
\end{aligned}$$

Newtons' iterations

The Newtons' iterations for solving $\mathcal{G}\mathbf{X} = \mathbf{0}$ is

$$\begin{aligned} \mathbf{G}_0 &= \mathbf{0} \qquad \text{for the minimal non-negative solution} \\ \mathbf{G}_{n+1} &= \mathbf{G_n} - \underbrace{\mathcal{G}'(\mathbf{G_n})^{-1}\mathcal{G}\mathbf{G_n}}_{\mathbf{X_n}} \ . \end{aligned}$$

That is $\mathbf{G}_{n+1} = \mathbf{G_n} - \mathbf{X_n}$ where $\mathbf{X_n}$ is the solution of

$$\mathcal{G}'(\mathbf{G_n})\mathbf{X_n} = \mathcal{G}\mathbf{G_n} \ ,$$

which is

$$\mathbf{L}\mathbf{X_n} + \mathbf{F}(\mathbf{G_n}\mathbf{X_n} + \mathbf{X_n}\mathbf{G_n}) = \mathbf{B} + \mathbf{L}\mathbf{G_n} + \mathbf{F}\mathbf{G_n}^2 \ .$$

In the last expression the unknown matrix, $\mathbf{X_n}$, is multiplied from both sides.

Newtons' iterations

An efficient way to solve

$$\underbrace{(\mathbf{L}+\mathbf{F}\mathbf{G_n})}_{\mathbf{E}}\mathbf{X_n}+\mathbf{F}\mathbf{X_n}\mathbf{G_n}=\underbrace{\mathbf{B}+\mathbf{L}\mathbf{G_n}+\mathbf{F}\mathbf{G_n}^2}_{\mathbf{C}} \ .$$

is via the real Schur decomposition of $\mathbf{G_n} = \mathbf{\Theta}'\mathbf{S_n}\mathbf{\Theta}$ where $\mathbf{S_n}$ is quasi upper-triangular and $\mathbf{\Theta}\mathbf{\Theta}' = \mathbf{\Theta}'\mathbf{\Theta} = \mathbf{I}$.

Let $\mathbf{V_n} = \mathbf{X_n}\mathbf{\Theta}'$ and multiply with $\mathbf{\Theta}'$ from the right then

$$\mathbf{E}\mathbf{V_n}+\mathbf{F}\mathbf{V_n}\mathbf{S_n}=\mathbf{C}\mathbf{\Theta}' \ .$$

Due to the quasi upper-triangular structure of $\mathbf{S_n}$ we can solve the matrix equation column-by-column.

For the first column we have

$$(\mathbf{E}+\mathbf{F}[\mathbf{S_n}]_{11})[\mathbf{V_n}]_1=\mathbf{C}[\mathbf{\Theta}']_1 \ .$$

Based on $[\mathbf{V_n}]_1$ we obtain a similar equation for the second column.

If there are complex eigenvalues $\mathbf{S_n}$ is not completely upper triangular, but there might be non-zero element in the first subdiagonal. In this case a linear system of two columns ($[\mathbf{V_n}]_{k-1}, [\mathbf{V_n}]_k$) needs to be solved.

Cyclic reduction

Cyclic reduction algorithm to calculate $\mathbf{G}$:

$$
\begin{aligned}
&\hat{\mathbf{L}}' := \mathbf{L};\\
&\mathbf{L}' := \mathbf{L};\\
&\mathbf{F}' := \mathbf{F};\\
&\mathbf{B}' := \mathbf{B};\\
&\mathbf{G} := \mathbf{0};\\
&\textbf{REPEAT}\\
&\quad \hat{\mathbf{L}}' := \hat{\mathbf{L}}' - \mathbf{F}'\mathbf{L}'^{-1}\mathbf{B}';\\
&\quad \mathbf{L}'' := \mathbf{L}' - \mathbf{F}'\mathbf{L}'^{-1}\mathbf{B}' - \mathbf{B}'\mathbf{L}'^{-1}\mathbf{F}';\\
&\quad \mathbf{F}' := -\mathbf{F}'\mathbf{L}'^{-1}\mathbf{F}';\\
&\quad \mathbf{B}' := -\mathbf{B}'\mathbf{L}'^{-1}\mathbf{B}';\\
&\quad \mathbf{L}' := \mathbf{L}'';\\
&\quad \mathbf{G}_{old} := \mathbf{G};\\
&\quad \mathbf{G} := -\hat{\mathbf{L}}'^{-1}\mathbf{B};\\
&\textbf{UNTIL} ||\mathbf{G} - \mathbf{G}_{old}|| \leq \epsilon
\end{aligned}
$$

Cyclic reduction

To solve

$$\mathbf{0} = \mathbf{B} + \mathbf{L}\mathbf{G} + \mathbf{F}\mathbf{G}^2$$

look for the solution of

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \end{array}$$

$$\begin{bmatrix} \mathbf{L} & \mathbf{F} & & & \\ \mathbf{B} & \mathbf{L} & \mathbf{F} & & \\ & \mathbf{B} & \mathbf{L} & \mathbf{F} & \\ & & \mathbf{B} & \mathbf{L} & \mathbf{F} \\ & & & \ddots & \ddots \end{bmatrix} \cdot \begin{bmatrix} \mathbf{G} \\ \mathbf{G}^2 \\ \mathbf{G}^3 \\ \mathbf{G}^4 \\ \vdots \end{bmatrix} = \begin{bmatrix} -\mathbf{B} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \end{bmatrix}$$

Cyclic reduction

After an odd-even permutation we have

$$
\begin{array}{cccc|cccc}
1 & 3 & 5 & \cdots & 2 & 4 & 6 & \cdots
\end{array}
$$

$$
\left[\begin{array}{cccc|cccc}
\hat{\mathbf{L}}_0 & & & & \mathbf{F} & & & \\
 & \mathbf{L} & & & \mathbf{B} & \mathbf{F} & & \\
 & & \mathbf{L} & & & \mathbf{B} & \mathbf{F} & \\
 & & & \ddots & & & \ddots & \ddots \\
\hline
\mathbf{B} & \mathbf{F} & & & \mathbf{L} & & & \\
 & \mathbf{B} & \mathbf{F} & & & \mathbf{L} & & \\
 & & \mathbf{B} & \ddots & & & \mathbf{L} & \\
 & & & \ddots & & & & \ddots
\end{array}\right]
\cdot
\left[\begin{array}{c}
\mathbf{G} \\ \mathbf{G}^3 \\ \mathbf{G}^5 \\ \vdots \\ \hline \mathbf{G}^2 \\ \mathbf{G}^4 \\ \mathbf{G}^6 \\ \vdots
\end{array}\right]
=
\left[\begin{array}{c}
-\mathbf{B} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \hline \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots
\end{array}\right]
$$

where $\hat{\mathbf{L}}_0 = \mathbf{L}$.

Denoting the parts by $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$, $\mathbf{A_4}$ we have

$$\mathbf{A_1 G}^{odd} + \mathbf{A_2 G}^{even} = \mathbf{B}^{odd} \ ,$$

$$\mathbf{A_3 G}^{odd} + \mathbf{A_4 G}^{even} = \mathbf{0} \ ,$$

from which

$$\left(\mathbf{A_1} - \mathbf{A_2}\mathbf{A_4}^{-1}\mathbf{A_3}\right) \mathbf{G}^{odd} = \mathbf{B}^{odd} \ .$$

Cyclic reduction

The obtained equation has the form

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & \cdots
\end{array}
$$

$$
\begin{bmatrix}
\hat{\mathbf{L}}_1 & \mathbf{F}_1 & & & \\
\mathbf{B}_1 & \mathbf{L}_1 & \mathbf{F}_1 & & \\
 & \mathbf{B}_1 & \mathbf{L}_1 & \mathbf{F}_1 & \\
 & & \mathbf{B}_1 & \mathbf{L}_1 & \mathbf{F}_1 \\
 & & & \ddots & \ddots
\end{bmatrix}
\cdot
\begin{bmatrix}
\mathbf{G} \\ \mathbf{G}^3 \\ \mathbf{G}^5 \\ \mathbf{G}^7 \\ \vdots
\end{bmatrix}
=
\begin{bmatrix}
-\mathbf{B} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots
\end{bmatrix}
,
$$

where

$$\mathbf{B}_1 = -\mathbf{B}_0\mathbf{L}_0{}^{-1}\mathbf{B}_0 \; , \mathbf{F}_1 = -\mathbf{F}_0\mathbf{L}_0{}^{-1}\mathbf{F}_0 \; ,$$

$$\mathbf{L}_1 = \mathbf{L}_0 - \mathbf{F}_0\mathbf{L}_0{}^{-1}\mathbf{B}_0 - \mathbf{B}_0\mathbf{L}_0{}^{-1}\mathbf{F}_0 \; ,$$

$$\hat{\mathbf{L}}_1 = \hat{\mathbf{L}}_0 - \mathbf{F}_0\mathbf{L}_0{}^{-1}\mathbf{B}_0 \; ,$$

with $\mathbf{B}_0 = \mathbf{B}$, $\mathbf{F}_0 = \mathbf{F}$, $\mathbf{L}_0 = \mathbf{L}$.

Cyclic reduction

Iteratively repeating the same shame we have

$$
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & \cdots \end{array} \\
\left[\begin{array}{ccccc}
\hat{\mathbf{L}}_\mathbf{n} & \mathbf{F}_\mathbf{n} & & & \\
\mathbf{B}_\mathbf{n} & \mathbf{L}_\mathbf{n} & \mathbf{F}_\mathbf{n} & & \\
& \mathbf{B}_\mathbf{n} & \mathbf{L}_\mathbf{n} & \mathbf{F}_\mathbf{n} & \\
& & \mathbf{B}_\mathbf{n} & \mathbf{L}_\mathbf{n} & \mathbf{F}_\mathbf{n} \\
& & & \ddots & \ddots
\end{array}\right]
\end{array}
\cdot
\left[\begin{array}{c}
\mathbf{G} \\
\mathbf{G}^{2^n+1} \\
\mathbf{G}^{2\cdot 2^n+1} \\
\mathbf{G}^{3\cdot 2^n+1} \\
\vdots
\end{array}\right]
=
\left[\begin{array}{c}
-\mathbf{B} \\
\mathbf{0} \\
\mathbf{0} \\
\mathbf{0} \\
\vdots
\end{array}\right],
$$

where, from the first row,

$$\mathbf{G} = \underbrace{-\hat{\mathbf{L}}_\mathbf{n}^{-1}\mathbf{B}}_{\mathbf{G}_\mathbf{n}} - \hat{\mathbf{L}}_\mathbf{n}^{-1}\mathbf{F}_\mathbf{n}\mathbf{G}^{2^n+1} \ .$$

$$\mathbf{G}_\mathbf{n} = -\hat{\mathbf{L}}_\mathbf{n}^{-1}\mathbf{B}$$

is the estimated $\mathbf{G}$ after n iterations.

Quasi birth-death process with irregular level 0

QBD with general level 0 (e.g., different size):

$\rightarrow$ irregular part: level 0, regular part: level $1, 2, \ldots$

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|}\hline \mathbf{L}' & \mathbf{F}' & & & \\ \hline \mathbf{B}' & \mathbf{L}'' & \mathbf{F} & & \\ \hline & \mathbf{B} & \mathbf{L} & \mathbf{F} & \\ \hline & & \mathbf{B} & \mathbf{L} & \mathbf{F} \\ \hline & & & \ddots & \ddots \\ \hline \end{array}$$

Linear system for $\boldsymbol{\pi}_0$ and $\boldsymbol{\pi}_1$:

$$[\boldsymbol{\pi}_0 | \boldsymbol{\pi}_1] \begin{array}{|c|c|}\hline \mathbf{L}' & \mathbf{F}' \\ \hline \mathbf{B}' & \mathbf{L}'' + \mathbf{R}\mathbf{B} \\ \hline \end{array} = [\ \mathbf{0} \mid \mathbf{0}\ ]$$

$$\boldsymbol{\pi}_0 \mathbb{1} + \boldsymbol{\pi}_1 (\mathbf{I} - \mathbf{R})^{-1} \mathbb{1} = 1$$

Quasi birth-death process with irregular level 0

Example: M/PH/1 queue

- arrival process: Poisson process with parameter λ,
- service time: PH distributed with representation $\tau, \mathbf{T}$. $(t = -\mathbf{T}\mathbb{1})$

Structure of the transition probability matrix:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|} \hline -\lambda & \lambda\tau & & & \\ \hline t & \mathbf{T}-\lambda\mathbf{I} & \lambda\mathbf{I} & & \\ \hline & t\tau & \mathbf{T}-\lambda\mathbf{I} & \lambda\mathbf{I} & \\ \hline & & t\tau & \mathbf{T}-\lambda\mathbf{I} & \lambda\mathbf{I} \\ \hline & & & \ddots & \ddots \\ \hline \end{array}$$

That is $\mathbf{F} = \lambda\mathbf{I}$, $\mathbf{L} = \mathbf{L}'' = \mathbf{T} - \lambda\mathbf{I}$, $\mathbf{B} = t\tau$ and $\mathbf{F}' = \lambda\tau$, $\mathbf{L}' = -\lambda$, $\mathbf{B}' = t$.

Quasi birth-death process with irregular level 0

Example: MAP/PH/1 queue

- arrival process: MAP with representation $\mathbf{D_0}, \mathbf{D_1}$,
- service time: PH distributed with representation $\tau, \mathbf{T}$. $(t = -\mathbf{T}\mathbb{1})$

Structure of the transition probability matrix:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|} \hline \mathbf{D_0} & \mathbf{D_1} \otimes \tau & & & \\ \hline \mathbf{I} \otimes t & \mathbf{D_0} \oplus \mathbf{T} & \mathbf{D_1} \otimes \mathbf{I} & & \\ \hline & \mathbf{I} \otimes t\tau & \mathbf{D_0} \oplus \mathbf{T} & \mathbf{D_1} \otimes \mathbf{I} & \\ \hline & & \mathbf{I} \otimes t\tau & \mathbf{D_0} \oplus \mathbf{T} & \mathbf{D_1} \otimes \mathbf{I} \\ \hline & & & \ddots & \ddots \\ \hline \end{array}$$

That is $\mathbf{F} = \mathbf{D_1} \otimes \mathbf{I}$, $\mathbf{L} = \mathbf{D_0} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{T} = \mathbf{D_0} \oplus \mathbf{T}$, $\mathbf{B} = \mathbf{I} \otimes t\tau$ and $\mathbf{F}' = \mathbf{D_1} \otimes \tau$, $\mathbf{L}' = \mathbf{D_0}$, $\mathbf{B}' = \mathbf{I} \otimes t$.

Finite quasi birth-death process

When the level process has an upper bound at level m the generator matrix takes the form:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|} \hline \mathbf{L}' & \mathbf{F} & & & \\ \hline \mathbf{B} & \mathbf{L} & \ddots & & \\ \hline & \mathbf{B} & \ddots & \mathbf{F} & \\ \hline & & \ddots & \mathbf{L} & \mathbf{F} \\ \hline & & & \mathbf{B} & \mathbf{L}'' \\ \hline \end{array}$$

Stationary equations:

$$\boldsymbol{\pi}_0 \mathbf{L}' + \boldsymbol{\pi}_1 \mathbf{B} = \mathbf{0}$$

$$\boldsymbol{\pi}_{n-1} \mathbf{F} + \boldsymbol{\pi}_n \mathbf{L} + \boldsymbol{\pi}_{n+1} \mathbf{B} = \mathbf{0} \qquad 1 \le n \le m-1$$

$$\boldsymbol{\pi}_{m-1} \mathbf{F} + \boldsymbol{\pi}_m \mathbf{L}'' = \mathbf{0}$$

$$\sum_{n=0}^{m} \boldsymbol{\pi}_n \mathbb{1} = 1$$

Finite quasi birth-death process

Due to the finite structure the stationary solution is not geometric.

Conjecture:

We assume that the solution is a linear combination of two geometric series starting from the two bounds of the level process. I.e.,

$$\boldsymbol{\pi}_n = \boldsymbol{\alpha}\mathbf{R}^n + \boldsymbol{\beta}\mathbf{S}^{m-n}, \quad \forall 0 \leq n \leq m,$$

where matrix $\mathbf{R}$ and $\mathbf{S}$ are the solution of the matrix equations:

$$\mathbf{F} + \mathbf{R}\mathbf{L} + \mathbf{R}^2\mathbf{B} = \mathbf{0}$$

$$\mathbf{B} + \mathbf{S}\mathbf{L} + \mathbf{S}^2\mathbf{F} = \mathbf{0}$$

If in the homogeneous part the drift is

- negative: $|\lambda_i(\mathbf{R})| < 1$ and $|\lambda_i(\mathbf{S})| \leq 1$,
- positive: $|\lambda_i(\mathbf{R})| \leq 1$ and $|\lambda_i(\mathbf{S})| < 1$,
- zero: $|\lambda_i(\mathbf{R})| \leq 1$ and $|\lambda_i(\mathbf{S})| \leq 1$.

Finite quasi birth-death process

The conjecture satisfies the equations:

$$\boldsymbol{\pi}_{n-1}\mathbf{F} + \boldsymbol{\pi}_n\mathbf{L} + \boldsymbol{\pi}_{n+1}\mathbf{B} = \mathbf{0} \qquad 1 \leq n \leq m-1$$

The unknown vectors, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, are obtained from the remaining equations as the solution of the linear system:

$$[\boldsymbol{\alpha} \mid \boldsymbol{\beta}] \left[\begin{array}{c|c} \mathbf{L}' + \mathbf{R}\mathbf{B} & \mathbf{R}^{m-1}\left(\mathbf{F} + \mathbf{R}\mathbf{L}''\right) \\ \hline \mathbf{S}^{m-1}\left(\mathbf{S}\mathbf{L}' + \mathbf{B}\right) & \mathbf{S}\mathbf{F} + \mathbf{L}'' \end{array}\right] = [\mathbf{0} \mid \mathbf{0}]$$

$$\boldsymbol{\alpha}\sum_{n=0}^{m}\mathbf{R}^n\mathbb{1} + \boldsymbol{\beta}\sum_{n=0}^{m}\mathbf{S}^n\mathbb{1} = 1$$

Finite quasi birth-death process

Computation of $\sum_{k=0}^{m} \mathbf{R}^k$ (and $\sum_{k=0}^{m} \mathbf{S}^k$):

- if $|\lambda_i(\mathbf{R})| < 1$, $\forall i \in (1, \ldots, n)$:

$$\sum_{k=0}^{m} \mathbf{R}^k = (\mathbf{I} - \mathbf{R}^{m+1})(\mathbf{I} - \mathbf{R})^{-1}$$

- if $|\lambda_i(\mathbf{R})| \leq 1$, such that $\lambda_1(\mathbf{R}) = 1$ and $|\lambda_i(\mathbf{R})| < 1$, $\forall i \in (2, \ldots, n)$:

$$\sum_{k=0}^{m} \mathbf{R}^k = \Big(\mathbf{I} - (\mathbf{R} - \mathbf{\Pi})^{m+1}\Big)\Big(\mathbf{I} - (\mathbf{R} - \mathbf{\Pi})\Big)^{-1} + m\mathbf{\Pi}$$

  where

$$\mathbf{\Pi} = \frac{uv}{vu} \ ,$$

  column vector u is a non-zero solution of

$$\mathbf{R}u = u$$

  and row vector v is a non-zero solution of

$$v\mathbf{R} = v \ .$$

  Note that $(\mathbf{R} - \mathbf{\Pi})\mathbf{\Pi} = \mathbf{\Pi}(\mathbf{R} - \mathbf{\Pi}) = \mathbf{0}$, $\mathbf{\Pi}^i = \mathbf{\Pi}$ and $\mathbf{R}^k = ((\mathbf{R} - \mathbf{\Pi}) + \mathbf{\Pi})^k = (\mathbf{R} - \mathbf{\Pi})^k + \underbrace{(\mathbf{R} - \mathbf{\Pi})\mathbf{\Pi} \ldots}_{\mathbf{0}} + \mathbf{\Pi}^i$

Piecewise constant infinite QBD with 2 parts

The process behaviour changes at level m:

$$\mathbf{Q} = \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \cdots & m{-}1 & m & m{+}1 & \cdots & \end{array} \\ \left[\begin{array}{cccccccc} \mathbf{L}' & \mathbf{F} & & & & & & \\ \mathbf{B} & \mathbf{L} & \ddots & & & & & \\ & \mathbf{B} & \ddots & \mathbf{F} & & & & \\ & & \ddots & \mathbf{L} & \mathbf{F} & & & \\ & & & \mathbf{B} & \mathbf{L}'' & \hat{\mathbf{F}} & & \\ & & & & \hat{\mathbf{B}} & \hat{\mathbf{L}} & \hat{\mathbf{F}} & \\ & & & & & \hat{\mathbf{B}} & \hat{\mathbf{L}} & \ddots \\ & & & & & & \ddots & \ddots \end{array}\right] \end{array}$$

Stationary equations:

$$\begin{aligned} \boldsymbol{\pi}_0\mathbf{L}' + \boldsymbol{\pi}_1\mathbf{B} &= \mathbf{0} \qquad && n = 0 \\ \boldsymbol{\pi}_{n-1}\mathbf{F} + \boldsymbol{\pi}_n\mathbf{L} + \boldsymbol{\pi}_{n+1}\mathbf{B} &= \mathbf{0} && 1 \le n \le m-1 \\ \boldsymbol{\pi}_{m-1}\mathbf{F} + \boldsymbol{\pi}_m\mathbf{L}'' + \boldsymbol{\pi}_{m+1}\hat{\mathbf{B}} &= \mathbf{0} && n = m \\ \boldsymbol{\pi}_{n-1}\hat{\mathbf{F}} + \boldsymbol{\pi}_n\hat{\mathbf{L}} + \boldsymbol{\pi}_{n+1}\hat{\mathbf{B}} &= \mathbf{0} && m+1 \le n \\ \sum_{n=0}^{\infty} \boldsymbol{\pi}_n \mathbb{1} &= 1 \end{aligned}$$

Piecewise constant infinite QBD with 2 parts

Conjecture:

From levels 0 to m the solution is a linear combination of two geometric series and from level m on it is matrix geometric.

$$\boldsymbol{\pi}_n = \boldsymbol{\alpha}\mathbf{R}^n + \boldsymbol{\beta}\mathbf{S}^{m-n}, \quad 0 \leq n \leq m,$$

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_m\hat{\mathbf{R}}^{n-m} = (\boldsymbol{\alpha}\mathbf{R}^m + \boldsymbol{\beta})\hat{\mathbf{R}}^{n-m}, \quad m < n,$$

where matrices $\mathbf{R}$, $\mathbf{S}$ and $\hat{\mathbf{R}}$ are the solution of the matrix equations:

$$\mathbf{F} + \mathbf{R}\mathbf{L} + \mathbf{R}^2\mathbf{B} = \mathbf{0}$$

$$\mathbf{B} + \mathbf{S}\mathbf{L} + \mathbf{S}^2\mathbf{F} = \mathbf{0}$$

$$\hat{\mathbf{F}} + \hat{\mathbf{R}}\hat{\mathbf{L}} + \hat{\mathbf{R}}^2\hat{\mathbf{B}} = \mathbf{0}$$

The conjecture satisfies the regular equations:

$$\boldsymbol{\pi}_{n-1}\mathbf{F} + \boldsymbol{\pi}_n\mathbf{L} + \boldsymbol{\pi}_{n+1}\mathbf{B} = \mathbf{0} \qquad 1 \leq n \leq m-1$$

$$\boldsymbol{\pi}_{n-1}\hat{\mathbf{F}} + \boldsymbol{\pi}_n\hat{\mathbf{L}} + \boldsymbol{\pi}_{n+1}\hat{\mathbf{B}} = \mathbf{0} \qquad m+1 \leq n$$

Piecewise constant infinite QBD with 2 parts

The unknown vectors, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, are obtained from the irregular equations (for level 0 and m) as the solution of the linear system:

$$[\boldsymbol{\alpha} \mid \boldsymbol{\beta}] \cdot$$

$\mathbf{L}' + \mathbf{R}\mathbf{B}$	$\mathbf{R}^{m-1}\left(\mathbf{F} + \mathbf{R}(\mathbf{L}'' + \hat{\mathbf{R}}\hat{\mathbf{B}})\right)$
$\mathbf{S}^{m-1}(\mathbf{S}\mathbf{L}' + \mathbf{B})$	$\mathbf{S}\mathbf{F} + \mathbf{L}'' + \hat{\mathbf{R}}\hat{\mathbf{B}}$

$$= [\mathbf{0} \mid \mathbf{0}]$$

$$\boldsymbol{\alpha} \sum_{n=0}^{m-1} \mathbf{R}^n \mathbb{1} + \boldsymbol{\beta} \sum_{n=0}^{m-1} \mathbf{S}^n \mathbb{1} + (\boldsymbol{\alpha}\mathbf{R}^m + \boldsymbol{\beta})(\mathbf{I} - \hat{\mathbf{R}})^{-1} \mathbb{1} = 1$$

Piecewise constant finite QBD with 2 parts

$$
\mathbf{Q} = \begin{array}{c|c|c|c|c|c|c|c|c|}
0 & 1 & \cdots & m{-}1 & m & m{+}1 & \cdots & M{-}1 & M \\
\hline
\mathbf{L}' & \mathbf{F} & & & & & & & \\
\hline
\mathbf{B} & \mathbf{L} & \ddots & & & & & & \\
\hline
 & \mathbf{B} & \ddots & \mathbf{F} & & & & & \\
\hline
 & & \ddots & \mathbf{L} & \mathbf{F} & & & & \\
\hline
 & & & \mathbf{B} & \mathbf{L}'' & \hat{\mathbf{F}} & & & \\
\hline
 & & & & \hat{\mathbf{B}} & \hat{\mathbf{L}} & \ddots & & \\
\hline
 & & & & & \hat{\mathbf{B}} & \ddots & \hat{\mathbf{F}} & \\
\hline
 & & & & & & \ddots & \hat{\mathbf{L}} & \hat{\mathbf{F}} \\
\hline
 & & & & & & & \hat{\mathbf{B}} & \mathbf{L}^* \\
\hline
\end{array}
$$

Stationary equations:

$$
\begin{aligned}
\boldsymbol{\pi}_0\mathbf{L}' + \boldsymbol{\pi}_1\mathbf{B} &= \mathbf{0} && n = 0 \\
\boldsymbol{\pi}_{n-1}\mathbf{F} + \boldsymbol{\pi}_n\mathbf{L} + \boldsymbol{\pi}_{n+1}\mathbf{B} &= \mathbf{0} && 0 < n < m \\
\boldsymbol{\pi}_{m-1}\mathbf{F} + \boldsymbol{\pi}_m\mathbf{L}'' + \boldsymbol{\pi}_{m+1}\hat{\mathbf{B}} &= \mathbf{0} && n = m \\
\boldsymbol{\pi}_{n-1}\hat{\mathbf{F}} + \boldsymbol{\pi}_n\hat{\mathbf{L}} + \boldsymbol{\pi}_{n+1}\hat{\mathbf{B}} &= \mathbf{0} && m < n < M \\
\boldsymbol{\pi}_{m-1}\mathbf{F} + \boldsymbol{\pi}_m\mathbf{L}^* + \boldsymbol{\pi}_{m+1}\hat{\mathbf{B}} &= \mathbf{0} && n = M \\
\sum_{n=0}^{\infty} \boldsymbol{\pi}_n \mathbb{1} &= 1
\end{aligned}
$$

Piecewise constant finite QBD with 2 parts

Conjecture:

$$\boldsymbol{\pi}_n = \boldsymbol{\alpha}\mathbf{R}^n + \boldsymbol{\beta}\mathbf{S}^{m-n}, \quad 0 \le n \le m,$$

$$\boldsymbol{\pi}_n = \boldsymbol{\gamma}\hat{\mathbf{R}}^{n-m} + \boldsymbol{\delta}\hat{\mathbf{S}}^{M-n}, \quad m \le n \le M,$$

where matrices $\mathbf{R}$, $\mathbf{S}$ and $\hat{\mathbf{R}}$, $\hat{\mathbf{S}}$ are the solution of the matrix equations:

$$\mathbf{F} + \mathbf{R}\mathbf{L} + \mathbf{R}^2\mathbf{B} = \mathbf{0}, \quad \mathbf{B} + \mathbf{S}\mathbf{L} + \mathbf{S}^2\mathbf{F} = \mathbf{0},$$

$$\hat{\mathbf{F}} + \hat{\mathbf{R}}\hat{\mathbf{L}} + \hat{\mathbf{R}}^2\hat{\mathbf{B}} = \mathbf{0}, \quad \hat{\mathbf{B}} + \hat{\mathbf{S}}\hat{\mathbf{L}} + \hat{\mathbf{S}}^2\hat{\mathbf{F}} = \mathbf{0}.$$

The conjecture satisfies the regular equations:

$$\boldsymbol{\pi}_{n-1}\mathbf{F} + \boldsymbol{\pi}_n\mathbf{L} + \boldsymbol{\pi}_{n+1}\mathbf{B} = \mathbf{0} \qquad 0 < n < m$$

$$\boldsymbol{\pi}_{n-1}\hat{\mathbf{F}} + \boldsymbol{\pi}_n\hat{\mathbf{L}} + \boldsymbol{\pi}_{n+1}\hat{\mathbf{B}} = \mathbf{0} \qquad m < n < M$$

Piecewise constant finite QBD with 2 parts

The unknown vectors, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$, are obtained from the set of linear equations composed by the irregular equations (for level 0, m, m, M), where the two boundary equations for level m utilizes the two different forms of $\boldsymbol{\pi}_m$.

$$[\boldsymbol{\alpha} \mid \boldsymbol{\beta} \mid \boldsymbol{\gamma} \mid \boldsymbol{\delta}] \cdot$$

$$\boxed{\begin{array}{cccc} \mathbf{L}' + \mathbf{R}\mathbf{B} & \mathbf{R}^{m-1}\left(\mathbf{F}+\mathbf{R}\mathbf{L}''\right) & \mathbf{R}^{m-1}\mathbf{F} & \mathbf{0} \\ \mathbf{S}^{m-1}(\mathbf{S}\mathbf{L}'+\mathbf{B}) & \mathbf{S}\mathbf{F} + \mathbf{L}'' & \mathbf{S}\mathbf{F} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{R}}\hat{\mathbf{B}} & \mathbf{L}'' + \hat{\mathbf{R}}\hat{\mathbf{B}} & \hat{\mathbf{R}}^{M-m-1}\left(\hat{\mathbf{F}}+\hat{\mathbf{R}}\mathbf{L}^*\right) \\ \mathbf{0} & \hat{\mathbf{S}}^{M-m-1}\hat{\mathbf{B}} & \hat{\mathbf{S}}^{M-m-1}\left(\hat{\mathbf{B}}+\hat{\mathbf{S}}\mathbf{L}''\right) & \hat{\mathbf{S}}\hat{\mathbf{F}} + \mathbf{L}^* \end{array}}$$

$$= [\mathbf{0} \mid \mathbf{0} \mid \mathbf{0} \mid \mathbf{0}]$$

$$\boldsymbol{\alpha} \sum_{n=0}^{m-1} \mathbf{R}^n \mathbb{1} + \boldsymbol{\beta} \sum_{n=0}^{m-1} \mathbf{S}^n \mathbb{1} + \boldsymbol{\gamma} \sum_{n=0}^{M-m} \hat{\mathbf{R}}^n \mathbb{1} + \boldsymbol{\delta} \sum_{n=0}^{M-m} \hat{\mathbf{S}}^n \mathbb{1} = 1$$

QBD with closed form solution

M/PH/1 queue:

Structure of the generator matrix:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|} \hline -\lambda & \lambda\boldsymbol{\alpha} & & & \\ \hline \mathbf{a} & \mathbf{A}-\lambda\mathbf{I} & \lambda\mathbf{I} & & \\ \hline & \mathbf{a}\boldsymbol{\alpha} & \mathbf{A}-\lambda\mathbf{I} & \lambda\mathbf{I} & \\ \hline & & \mathbf{a}\boldsymbol{\alpha} & \mathbf{A}-\lambda\mathbf{I} & \lambda\mathbf{I} \\ \hline & & & \ddots & \ddots \\ \hline \end{array}$$

Stationary solution: $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$, $\boldsymbol{\pi}\mathbb{1} = 1$,

where $\boldsymbol{\pi} = \{\pi_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots\}$.

Utilization: $\rho = 1 - \pi_0 = \lambda\ E(PH) = \lambda\ \boldsymbol{\alpha}(-\mathbf{A})^{-1}\mathbb{1}$

QBD with closed form solution

M/PH/1 queue balance equations:

$$-\pi_0\lambda + \boldsymbol{\pi}_1\mathbf{a} = 0 \qquad (*1)$$

$$\pi_0\lambda\boldsymbol{\alpha} + \boldsymbol{\pi}_1(\mathbf{A}–\lambda\mathbf{I}) + \boldsymbol{\pi}_2\mathbf{a}\boldsymbol{\alpha} = \mathbf{0} \qquad (*2)$$

$$\boldsymbol{\pi}_{n-1}\lambda\mathbf{I} + \boldsymbol{\pi}_n(\mathbf{A}–\lambda\mathbf{I}) + \boldsymbol{\pi}_{n+1}\mathbf{a}\boldsymbol{\alpha} = \mathbf{0} \qquad \forall n \geq 2 \quad (*3)$$

First we show that

$$\lambda\boldsymbol{\pi}_n\mathbb{1} = \boldsymbol{\pi}_{n+1}\mathbf{a} \qquad \forall n \geq 1. \quad (*4)$$

Substituting $(*1)$ it into $(*2)$ gives:

$$\boldsymbol{\pi}_1(\mathbf{a}\boldsymbol{\alpha} + \mathbf{A} - \lambda\mathbf{I}) + \boldsymbol{\pi}_2\mathbf{a}\boldsymbol{\alpha} = \mathbf{0}$$

Multiplying this with $\mathbb{1}$ from the right gives $\boldsymbol{\pi}_1\lambda\mathbb{1} = \boldsymbol{\pi}_2\mathbf{a}$. Recursively substituting the result of the previous step and multiplying $(*3)$ with $\mathbb{1}$ results in $(*4)$.

Substituting $(*4)$ into $(*3)$ gives:

$$\lambda\boldsymbol{\pi}_{n-1} + \boldsymbol{\pi}_n(\mathbf{A}–\lambda\mathbf{I}) + \lambda\boldsymbol{\pi}_n\mathbb{1}\boldsymbol{\alpha} = \mathbf{0} \qquad \forall n \geq 2$$

and consequently

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_{n-1}\underbrace{\lambda(\lambda\mathbf{I} - \mathbf{A} - \lambda\mathbb{1}\boldsymbol{\alpha})^{-1}}_{\mathbf{R}} \qquad \forall n \geq 2.$$

From $(*2)$ we also have $\boldsymbol{\pi}_1 = \pi_0\boldsymbol{\alpha}\mathbf{R}$.

$\longrightarrow$ matrix geometric distribution:

$$\boldsymbol{\pi}_n = (1-\rho)\boldsymbol{\alpha}\mathbf{R}^n \qquad \forall n \geq 1.$$

QBD with "closed" form solution

PH/M/1 queue:

Structure of the generator matrix:

$$
\mathbf{Q} = \begin{array}{|c|c|c|c|c|}
\hline
\mathbf{A} & \mathbf{a}\boldsymbol{\alpha} & & & \\
\hline
\mu\mathbf{I} & \mathbf{A}-\mu\mathbf{I} & \mathbf{a}\boldsymbol{\alpha} & & \\
\hline
 & \mu\mathbf{I} & \mathbf{A}-\mu\mathbf{I} & \mathbf{a}\boldsymbol{\alpha} & \\
\hline
 & & \mu\mathbf{I} & \mathbf{A}-\lambda\mathbf{I} & \mathbf{a}\boldsymbol{\alpha} \\
\hline
 & & & \ddots & \ddots \\
\hline
\end{array}
$$

Stationary solution: $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$, $\boldsymbol{\pi}\mathbb{1} = 1$,

where $\boldsymbol{\pi} = \{\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots\}$.

Utilization: $\rho = 1 - \boldsymbol{\pi}_0\mathbb{1} = \dfrac{1}{\mu\ E(PH)} = \dfrac{1}{\mu\ \boldsymbol{\alpha}(-\mathbf{A})^{-1}\mathbb{1}}$

QBD with "closed" form solution

PH/M/1 queue balance equations:

$$\boldsymbol{\pi}_0\mathbf{A} + \boldsymbol{\pi}_1\mu\mathbf{I} = \mathbf{0} \qquad (*)$$

$$\boldsymbol{\pi}_{n-1}\mathbf{a}\boldsymbol{\alpha} + \boldsymbol{\pi}_n(\mathbf{A}–\mu\mathbf{I}) + \boldsymbol{\pi}_{n+1}\mu\mathbf{I} = \mathbf{0} \qquad \forall n \geq 1 \quad (**)$$

From $(*)$ we have $\boldsymbol{\pi}_0 = \mu\boldsymbol{\pi}_1(-\mathbf{A})^{-1}$.

The form of the stationary matrix geometric solution is $\boldsymbol{\pi}_n = \boldsymbol{\pi}_0\mathbf{R}^n$ where matrix $\mathbf{R}$ satisfies the matrix equation:

$$\mathbf{a}\boldsymbol{\alpha} + \mathbf{R}(\mathbf{A}–\mu\mathbf{I}) + \mathbf{R}^2\mu = \mathbf{0}.$$

Due to the fact that the first term, $\boldsymbol{a}\boldsymbol{\alpha}$, is a diad matrix $\mathbf{R}$ is a diad as well in the form $\mathbf{R} = \boldsymbol{a}\boldsymbol{r}$, where $\boldsymbol{r}$ is an unknown row vector. This diadic form results that $\boldsymbol{r}$ is proportional with $\boldsymbol{\pi}_n, \forall n \geq 1$.

From $(*)$ and $\boldsymbol{\pi}_0 = \mu\boldsymbol{\pi}_1(-\mathbf{A})^{-1}$ we also have

$$\underbrace{\boldsymbol{\pi}_0\mathbf{A}}_{-\mu\boldsymbol{\pi}_1} = -\mu\underbrace{\boldsymbol{\pi}_0\boldsymbol{a}}_{\mu\boldsymbol{\pi}_1\mathbb{1}}\boldsymbol{r}$$

and

$$\boldsymbol{r} = \frac{\boldsymbol{\pi}_1}{\mu\boldsymbol{\pi}_1\mathbb{1}}.$$

QBD with "closed" form solution

Substituting these into $(**)$ with $n = 1$ gives:

$$\mu\boldsymbol{\pi}_1 \mathbb{1}\boldsymbol{\alpha} + \boldsymbol{\pi}_1(\mathbf{A}–\mu\mathbf{I}) + \mu\boldsymbol{\pi}_1\boldsymbol{a}\boldsymbol{r} = \mathbf{0}$$

and

$$\boldsymbol{\pi}_1 (\mu\mathbb{1}\boldsymbol{\alpha} + \mathbf{A} - \mu\mathbf{I}) = -\mu\boldsymbol{\pi}_1\boldsymbol{a}\boldsymbol{r} = \frac{-\boldsymbol{\pi}_1\boldsymbol{a}}{\boldsymbol{\pi}_1\mathbb{1}}\ \boldsymbol{\pi}_1.$$

That is, $\boldsymbol{\pi}_1$ is the left eigenvector of $(\mu\mathbb{1}\boldsymbol{\alpha} + \mathbf{A} - \mu\mathbf{I})$ whose associated eigenvalue is the coefficient on the right hand side.

From $\boldsymbol{\pi}_n = \boldsymbol{\pi}_1\mathbf{R}^{n-1}$ we have

$$\boldsymbol{\pi}_2 = \boldsymbol{\pi}_1\boldsymbol{a}\boldsymbol{r} = \frac{\boldsymbol{\pi}_1\boldsymbol{a}}{\mu\boldsymbol{\pi}_1\mathbb{1}}\ \boldsymbol{\pi}_1 \quad \text{and} \quad \boldsymbol{\pi}_n = c^{n-1}\ \boldsymbol{\pi}_1,$$

where $c = \boldsymbol{\pi}_1\boldsymbol{a}/\mu\boldsymbol{\pi}_1\mathbb{1}$.

From $\sum_n \boldsymbol{\pi}_n\mathbb{1} = 1$ the normalizing condition for $\boldsymbol{\pi}_1$ is

$$\boldsymbol{\pi}_1 \left(\mu(-\mathbf{A})^{-1}\mathbb{1} + \frac{1}{1-c}\mathbb{1}\right) = 1\ .$$

QBD with "closed" form solution

On the other hand, multiplying $(*)$ with $\mathbb{1}$ from the right results

$$\boldsymbol{\pi}_0 \mathbf{a} = \boldsymbol{\pi}_1 \mu \mathbb{1}.$$

Recursively multiplying $(**)$ with $\mathbb{1}$ and substituting the previous result gives

$$\boldsymbol{\pi}_{n-1} \mathbf{a} = \boldsymbol{\pi}_n \mu \mathbb{1} \quad \forall n \geq 1.$$

Substituting this into $(**)$ we have:

$$\boldsymbol{\pi}_n \mu \mathbb{1} \boldsymbol{\alpha} + \boldsymbol{\pi}_n (\mathbf{A} - \mu \mathbf{I}) + \boldsymbol{\pi}_{n+1} \mu \mathbf{I} = \mathbf{0} \qquad \forall n \geq 1$$

hence

$$\boldsymbol{\pi}_{n+1} = \boldsymbol{\pi}_n \underbrace{(\mathbf{I} - \mathbf{A}/\mu - \mathbb{1}\boldsymbol{\alpha})}_{\text{this is not } \mathbf{R}!} \qquad \forall n \geq 1.$$

This relation does not hold for $n = 0$, but allows to compute, e.g.

$$\rho = \sum_{n=1}^{\infty} \boldsymbol{\pi}_1 (\mathbf{I} - \mathbf{A}/\mu - \mathbb{1}\boldsymbol{\alpha})^{n-1} \mathbb{1} = \boldsymbol{\pi}_1 (\mathbf{A}/\mu + \mathbb{1}\boldsymbol{\alpha})^{-1} \mathbb{1}$$

in closed form based on $\boldsymbol{\pi}_1$.

Inhomogeneous Quasi birth-death process

The transition rates (as well as level sizes) are level dependent:

- $\mathbf{F_n}$ – (forward) transitions from level n to $n+1$
- $\mathbf{L_n}$ – (local) transitions in level n
- $\mathbf{B_n}$ – (backward) transitions from level n to $n-1$.

Structure of the generator matrix:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|} \hline \mathbf{L_0} & \mathbf{F_0} & & & \\ \hline \mathbf{B_1} & \mathbf{L_1} & \mathbf{F_1} & & \\ \hline & \mathbf{B_2} & \mathbf{L_2} & \mathbf{F_2} & \\ \hline & & \mathbf{B_3} & \mathbf{L_3} & \mathbf{F_3} \\ \hline & & & \ddots & \ddots \\ \hline \end{array}$$

On the block level it still has a birth-death structure, but with level dependent rates.

The stationary equations are

$$0 = \pi_0 \mathbf{L_0} + \pi_1 \mathbf{B_1}$$

$$0 = \pi_{n-1} \mathbf{F_{n-1}} + \pi_n \mathbf{L_n} + \pi_{n+1} \mathbf{B_{n+1}} \text{ for } n \geq 1$$

Inhomogeneous Quasi birth-death process

The level dependent characteristic matrices are:

- $\mathbf{R_n}$ – "ratio of time spent in level n and $n+1$"
- $\mathbf{G_n}$ – "return probability from level n to level $n-1$"
- $\mathbf{U_n}$ – "generator of the restricted process on level n".

The stationary distribution has the form: $\pi_{n+1} = \pi_n \mathbf{R_n}$

With this form the stationary equations become

$$0 = \pi_0 \left(\mathbf{L_0} + \mathbf{R_0 B_1}\right)$$

$$0 = \pi_{n-1} \left(\mathbf{F_{n-1}} + \mathbf{R_{n-1} L_n} + \mathbf{R_{n-1} R_n B_{n+1}}\right) \text{ for } n \geq 1$$

The level dependent analysis of the characteristic matrices gives

$$\mathbf{0 = F_{n-1} + R_{n-1} L_n + R_{n-1} R_n B_{n+1}}$$

$$\mathbf{0 = B_n + L_n G_n + F_n G_{n+1} G_n}$$

$$\mathbf{U_n = L_n + F_n (-U_{n+1})^{-1} B_{n+1}}$$

Inhomogeneous Quasi birth-death process

Relation of the level dependent characteristic matrices

$$\mathbf{U_n} = \mathbf{L_n} + \mathbf{R_n B_{n+1}}$$
$$= \mathbf{L_n} + \mathbf{F_n G_{n+1}}$$

$$\mathbf{G_n} = (-\mathbf{U_n})^{-1}\mathbf{B_n}$$
$$= (-\mathbf{L_n} - \mathbf{R_n B_{n+1}})^{-1}\mathbf{B_n}$$

$$\mathbf{R_{n-1}} = \mathbf{F_{n-1}}(-\mathbf{U_n})^{-1}$$
$$= \mathbf{F_{n-1}}(-\mathbf{L_n} - \mathbf{F_{n+1} G_{n+1}})^{-1}$$

Numerical solution:

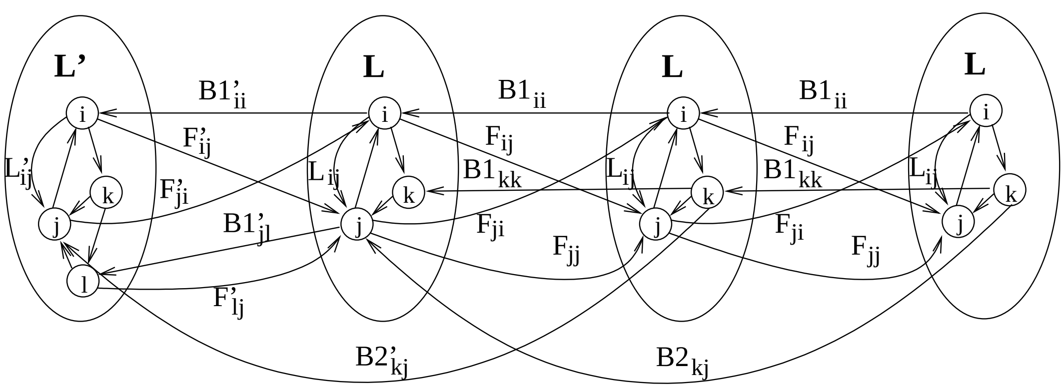
- start from a high level N,
  assuming $\mathbf{R_N} = \mathbf{R_{N-1}}$ (or $\mathbf{G_N} = \mathbf{G_{N+1}}$),
  and solve the quadratic equation for $\mathbf{R_N}$ (or $\mathbf{G_N}$),
- iteratively compute $\mathbf{R_n}$ from $\mathbf{R_{N-1}}$ to $\mathbf{R_0}$,
- obtain π_0 from $\pi_0(\mathbf{L_0} + \mathbf{R_0 B_1}) = 0$
  and $\pi_0 \sum_{n=0}^{N} \prod_{i=0}^{n-1} \mathbf{R_i} \mathbb{1} = 1$.

G/M/1-type process

$\{N(t), J(t)\}$ is a CTMC, where

- $N(t)$ is the "level" process (e.g., number of customers in a queue),
- $J(t)$ is the "phase" process (e.g., state of the environment).

$\{N(t), J(t)\}$ is a G/M/1-type process if upward transitions are restricted to one level up and there is no limit on downward transitions.



Level 0 is irregular (e.g., no departure).

G/M/1-type process

Notation

- $\mathbf{F}$ – transitions one level up (e.g., arrival)
- $\mathbf{L}$ – transitions in the same level
- $\mathbf{B_n}$ – transitions n level down (e.g., departure)
- $\mathbf{F}'$ – irregular block from level 0 to level 1.
- $\mathbf{L}'$ – irregular block at level 0.
- $\mathbf{B}'_\mathbf{n}$ – irregular blocks down to level 0

Structure of the transition probability matrix:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|} \hline \mathbf{L}' & \mathbf{F}' & & & \\ \hline \mathbf{B}'_1 & \mathbf{L} & \mathbf{F} & & \\ \hline \mathbf{B}'_2 & \mathbf{B}_1 & \mathbf{L} & \mathbf{F} & \\ \hline \mathbf{B}'_3 & \mathbf{B}_2 & \mathbf{B}_1 & \mathbf{L} & \mathbf{F} \\ \hline \vdots & & \ddots & \ddots & \ddots \\ \hline \end{array}$$

On the block level it has a G/M/1-type structure.

Condition of stability (G/M/1-type)

Asymptotically ($n \to \infty$) the phase process is a CTMC with generator matrix:

$$\mathbf{A} = \mathbf{F} + \mathbf{L} + \sum_{i=1}^{\infty} \mathbf{B_i}$$

Assuming $\mathbf{A}$ is irreducible, the stationary solution of $\mathbf{A}$ is:

$$\boldsymbol{\alpha}\mathbf{A} = \mathbf{0}, \boldsymbol{\alpha}\mathbb{1} = 1$$

The stationary drift of the level process is:

$$d = \boldsymbol{\alpha}\mathbf{F}\mathbb{1} - \boldsymbol{\alpha}\sum_{i=1}^{\infty} i\ \mathbf{B_i}\ \mathbb{1}$$

Condition of stability:

$$d = \boldsymbol{\alpha}\mathbf{F}\mathbb{1} - \boldsymbol{\alpha}\sum_{i=1}^{\infty} i\ \mathbf{B_i}\ \mathbb{1} < 0$$

Matrix geometric distribution (G/M/1-type)

Stationary solution: $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$, $\boldsymbol{\pi}\mathbb{1} = 1$.

Partitioning $\boldsymbol{\pi}$: $\boldsymbol{\pi} = \{\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots\}$

Decomposed stationary equations:

$$\boldsymbol{\pi}_0\mathbf{L}' + \sum_{i=1}^{\infty} \boldsymbol{\pi}_i\mathbf{B}'_\mathbf{i} = \mathbf{0}$$

$$\boldsymbol{\pi}_0\mathbf{F}' + \boldsymbol{\pi}_1\mathbf{L} + \sum_{i=1}^{\infty} \boldsymbol{\pi}_{i+1}\mathbf{B}_\mathbf{i} = \mathbf{0}$$

$$\boldsymbol{\pi}_{n-1}\mathbf{F} + \boldsymbol{\pi}_n\mathbf{L} + \sum_{i=1}^{\infty} \boldsymbol{\pi}_{n+i}\mathbf{B}_\mathbf{i} = \mathbf{0} \qquad \forall n \geq 2$$

$$\sum_{n=0}^{\infty} \boldsymbol{\pi}_n\mathbb{1} = 1$$

Conjecture: $\boldsymbol{\pi}_n = \boldsymbol{\pi}_{n-1}\mathbf{R}, \quad \forall n \geq 1 \qquad \rightarrow \qquad \boldsymbol{\pi}_n = \boldsymbol{\pi}_1\mathbf{R}^{n-1}$

where, matrix $\mathbf{R}$ is the solution of the matrix equation:

$$\mathbf{F} + \mathbf{R}\mathbf{L} + \sum_{i=1}^{\infty} \mathbf{R}^{i+1}\mathbf{B}_\mathbf{i} = \mathbf{0}$$

Matrix geometric distribution (G/M/1-type)

The conjecture satisfies the equations:

$$\boldsymbol{\pi}_{n-1}\mathbf{F} + \boldsymbol{\pi}_n\mathbf{L} + \sum_{i=1}^{\infty}\boldsymbol{\pi}_{n+i}\mathbf{B_i} = \mathbf{0} \qquad \forall n \geq 2$$

The remaining unknowns, $\boldsymbol{\pi}_0$ and $\boldsymbol{\pi}_1$, are the solution of the linear system:

$$[\boldsymbol{\pi}_0|\boldsymbol{\pi}_1] \left[\begin{array}{|c|c|} \hline \mathbf{L}' & \mathbf{F}' \\ \hline \sum_{i=1}^{\infty}\mathbf{R}^{i-1}\mathbf{B}'_\mathbf{i} & \mathbf{L} + \sum_{i=1}^{\infty}\mathbf{R}^{i}\mathbf{B_i} \\ \hline \end{array}\right] = [\ \mathbf{0}\ |\ \mathbf{0}\ ]$$

$$\boldsymbol{\pi}_0\mathbb{1} + \boldsymbol{\pi}_1(\mathbf{I}-\mathbf{R})^{-1}\mathbb{1} = 1$$

Matrix geometric distribution (G/M/1-type)

Linear algorithm to calculate $\mathbf{R}$:

$$\begin{array}{l} \mathbf{R} := \mathbf{0}; \\ \textbf{REPEAT} \\ \quad \mathbf{R}_{old} := \mathbf{R}; \\ \quad \mathbf{R} := \mathbf{F}\left(-\mathbf{L} - \sum_{i=1}^{\infty} \mathbf{R}^i \mathbf{B_i}\right)^{-1}; \\ \textbf{UNTIL} ||\mathbf{R} - \mathbf{R}_{old}|| \leq \epsilon \end{array}$$

Linear algorithm to calculate $\mathbf{R}$:

$$\begin{array}{l} \mathbf{R} := \mathbf{0}; \\ \textbf{REPEAT} \\ \quad \mathbf{R}_{old} := \mathbf{R}; \\ \quad \mathbf{R} := \left(-\mathbf{F} - \sum_{i=1}^{\infty} \mathbf{R}^{i+1} \mathbf{B_i}\right) \mathbf{L}^{-1}; \\ \textbf{UNTIL} ||\mathbf{R} - \mathbf{R}_{old}|| \leq \epsilon \end{array}$$

Matrix geometric distribution (G/M/1-type)

Properties of $\mathbf{R}$:

- the matrix equation has more than one solution.
- if the G/M/1-type process is stable there is a solution $\mathbf{R}$ whose eigenvalues ($\lambda_i(\mathbf{R})$) are $|\lambda_i(\mathbf{R})| < 1$ and this is the relevant $\mathbf{R}$ matrix.
- (if the G/M/1-type process is not stable there is a solution $\mathbf{R}$ whose eigenvalues ($\lambda_i(\mathbf{R})$) are $|\lambda_i(\mathbf{R})| \leq 1$ and this is the relevant $\mathbf{R}$ matrix.)

Stochastic interpretation:

$\mathbf{R}_{ij}$ is the ratio of the mean time spent in (n, j) and the mean time spent in $(n-1, i)$ before the first return to level $n-1$ starting from $(n-1, i)$.

In a homogeneous G/M/1-type process $\mathbf{R}_{ij}$ is independent of n.

Matrix geometric distribution (G/M/1-type)

Properties of the level crossing process:

- Matrix $\mathbf{G}$ cannot be used, because it is level dependent.

- Matrix $\mathbf{U}$, remains level independent.

Interpretation of $\mathbf{U}$:

The transient generator of the Markov chain restricted to level n before the first visit to level $n-1$.

Consequently $-\mathbf{U}^{-1}$ is the mean time spent in level n before the first visit to level $n-1$.

$\mathbf{U}$ satisfies:

$$\mathbf{U} = \mathbf{L} + \sum_{i=1}^{\infty} \left(\mathbf{F}(-\mathbf{U})^{-1}\right)^{i} \mathbf{B_i} = \mathbf{L} + \sum_{i=1}^{\infty} \mathbf{R}^{i}\mathbf{B_i}.$$

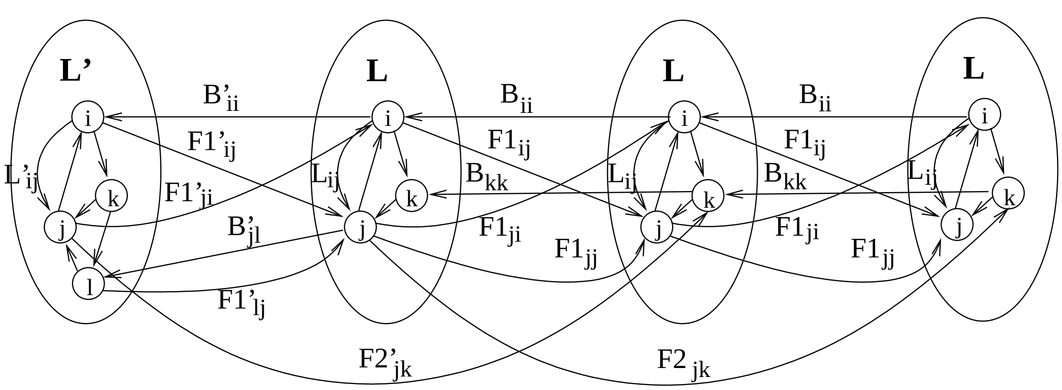
M/G/1-type process

$\{N(t), J(t)\}$ is a CTMC, where

- $N(t)$ is the "level" process (e.g., number of customers in a queue),

- $J(t)$ is the "phase" process (e.g., state of the environment).

$\{N(t), J(t)\}$ is an M/G/1-type process if downward transitions are restricted to one level down and there is no limit on upward transitions.



M/G/1-type process

Notation

- $\mathbf{L}$ – transitions in the same level
- $\mathbf{B}$ – transitions one level down (e.g., departure)
- $\mathbf{F_n}$ – transitions n level up (e.g., arrival)
- $\mathbf{L}'$ – irregular block at level 0.
- $\mathbf{B}'$ – irregular block from level 1 to level 0.
- $\mathbf{F_n'}$ – irregular blocks starting from level 0

Structure of the transition probability matrix:

$$\mathbf{Q} = \begin{array}{|c|c|c|c|c|} \hline \mathbf{L}' & \mathbf{F}'_1 & \mathbf{F}'_2 & \mathbf{F}'_3 & \mathbf{F}'_4 \\ \hline \mathbf{B}' & \mathbf{L} & \mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3 \\ \hline & \mathbf{B} & \mathbf{L} & \mathbf{F}_1 & \mathbf{F}_2 \\ \hline & & \mathbf{B} & \mathbf{L} & \mathbf{F}_1 \\ \hline & & & \ddots & \ddots \\ \hline \end{array}$$

On the block level it has an M/G/1-type structure.

Condition of stability (M/G/1-type)

Asymptotically ($n \to \infty$) the phase process is a CTMC with generator matrix:

$$\mathbf{A} = \mathbf{B} + \mathbf{L} + \sum_{i=1}^{\infty} \mathbf{F_i}$$

Assuming $\mathbf{A}$ is irreducible, the stationary solution of $\mathbf{A}$ is:

$$\boldsymbol{\alpha}\mathbf{A} = \mathbf{0}, \boldsymbol{\alpha}\mathbb{1} = 1$$

The stationary drift of the level process is:

$$d = \boldsymbol{\alpha} \sum_{i=1}^{\infty} i \ \mathbf{F_i}\mathbb{1} - \boldsymbol{\alpha}\mathbf{B} \ \mathbb{1}$$

Condition of stability:

$$d < 0$$

Stationary solution of M/G/1-type process

Stationary solution: $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$, $\boldsymbol{\pi}\mathbb{1} = 1$.

Decomposed stationary equations:

$$\boldsymbol{\pi}_0\mathbf{L}' + \boldsymbol{\pi}_1\mathbf{B}' = \mathbf{0}$$

$$\boldsymbol{\pi}_0\mathbf{F}'_\mathbf{n} + \sum_{i=1}^{n-1}\boldsymbol{\pi}_i\mathbf{F}_{\mathbf{n-i}} + \boldsymbol{\pi}_n\mathbf{L} + \boldsymbol{\pi}_{n+1}\mathbf{B} = \mathbf{0} \qquad \forall n \geq 1$$

$$\sum_{n=0}^{\infty}\boldsymbol{\pi}_n\mathbb{1} = 1$$

Inhomogeneous dependency structure
$\rightarrow$ non-geometric solution

Invariant metric of the level process:
matrix $\mathbf{G}$ (fundamental matrix)

$$\mathbf{B} + \mathbf{L}\mathbf{G} + \sum_{i=1}^{\infty}\mathbf{F}_\mathbf{i}\mathbf{G}^{i+1} = \mathbf{0}$$

Stationary solution of M/G/1-type process

Properties of $\mathbf{G}$:

- the matrix equation has more than one solution.
- if the M/G/1-type process is stable $\mathbf{G}$ is a stochastic matrix,
- (if the M/G/1-type process is transient $\mathbf{G}$ is a sub-stochastic matrix.)

Stochastic interpretation:

$\mathbf{G}_{ij}$ is the probability that starting from (n, i) the first state visited in level $n-1$ is $(n-1, j)$.

In a homogeneous M/G/1-type process $\mathbf{G}_{ij}$ is independent of n.

(Matrix $\mathbf{R}$ cannot be used.)

(If $\mathbf{B} = \boldsymbol{\gamma}^T \cdot \boldsymbol{\nu}$, where $\boldsymbol{\nu}\mathbb{1} = 1$, then $\mathbf{G} = \mathbb{1} \cdot \boldsymbol{\nu}$.)

Matrix $\mathbf{U}$ satisfies

$$\mathbf{U} = \mathbf{L} + \sum_{i=1}^{\infty} \mathbf{F_i} \Big(\underbrace{(-\mathbf{U})^{-1}\mathbf{B}}_{\mathbf{G}} \Big)^{i}$$

Stationary solution of M/G/1-type process

Linear algorithm to calculate $\mathbf{G}$:

$$
\begin{array}{l}
\mathbf{G} := \mathbf{I}; \\
\textbf{REPEAT} \\
\quad \mathbf{G}_{old} := \mathbf{G}; \\
\quad \mathbf{G} := \left(-\mathbf{L} - \sum_{i=1}^{\infty} \mathbf{F_i} \mathbf{G}^i \right)^{-1} \mathbf{B}; \\
\textbf{UNTIL} ||\mathbf{G} - \mathbf{G}_{old}|| \leq \epsilon
\end{array}
$$

Linear algorithm to calculate $\mathbf{G}$:

$$
\begin{array}{l}
\mathbf{G} := \mathbf{I}; \\
\textbf{REPEAT} \\
\quad \mathbf{G}_{old} := \mathbf{G}; \\
\quad \mathbf{G} := \mathbf{L}^{-1} \left(-\mathbf{B} - \sum_{i=1}^{\infty} \mathbf{F_i} \mathbf{G}^{i+1} \right); \\
\textbf{UNTIL} ||\mathbf{G} - \mathbf{G}_{old}|| \leq \epsilon
\end{array}
$$

Stationary solution of M/G/1-type process

Non-geometric solution $\rightarrow$ iterative computation of $\boldsymbol{\pi}_i$:

Ramaswami proposed the following one:

$$\boldsymbol{\pi}_i = -\left(\boldsymbol{\pi}_0 \mathbf{S'_i} + \sum_{k=1}^{i-1} \boldsymbol{\pi}_k \mathbf{S_{i-k}}\right) \mathbf{S_0}^{-1}, \quad \forall i \geq 1,$$

where for $i \geq 1$

$$\mathbf{S'_i} = \sum_{k=i}^{\infty} \mathbf{F'_k} \mathbf{G^{k-i}}, \ \mathbf{S_i} = \sum_{k=i}^{\infty} \mathbf{F_k} \mathbf{G^{k-i}} \quad \text{and} \quad \mathbf{S_0} = \mathbf{L} + \mathbf{S_1} \mathbf{G}.$$

The initial $\boldsymbol{\pi}_0$ vector is the solution of the linear system:

$$\boldsymbol{\pi}_0 \cdot \left(\mathbf{L'} - \mathbf{S'_1} (\mathbf{S_0})^{-1} \mathbf{B'}\right) = \mathbf{0}$$

$$\boldsymbol{\pi}_0 \mathbb{1} - \boldsymbol{\pi}_0 \sum_{i=1}^{\infty} \mathbf{S'_i} \left(\sum_{j=0}^{\infty} \mathbf{S_j}\right)^{-1} \mathbb{1} = 1$$

Stationary solution of M/G/1-type process

Let us consider the restricted process on level 0 and 1:

$$\mathbf{Q}^{(0,1)} = \begin{array}{|c|c|} \hline \mathbf{L}' & \mathbf{S}'_1 \\ \hline \mathbf{B}' & \mathbf{S_0} \\ \hline \end{array} \ ,$$

where $\mathbf{S}'_1$ contains all possible transitions from level 0 to level 1, $\mathbf{S}'_1 = \sum_{k=1}^{\infty} \mathbf{F}'_k \mathbf{G}^{k-1}$, and $\mathbf{S_0} = \mathbf{U} = \mathbf{L} + \sum_{i=1}^{\infty} \mathbf{F_i} \mathbf{G}^i$.

Further restricting the process to level 0,

$$\mathbf{Q}^{(0)} = \mathbf{L}' + \mathbf{S}'_1(-\mathbf{S_0})^{-1}\mathbf{B}' \ ,$$

from which

$$\pi_0(\mathbf{L}' + \mathbf{S}'_1(-\mathbf{S_0})^{-1}\mathbf{B}') = 0.$$

From $(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1)\mathbf{Q}^{(0,1)} = 0$ we have

$$\boldsymbol{\pi}_1 = \boldsymbol{\pi}_0 \mathbf{S}'_1 (-\mathbf{S_0})^{-1} \ .$$

Stationary solution of M/G/1-type process

Similarly, let us consider the restricted process on level 0, 1 and 2:

$$\mathbf{Q}^{(0,1,2)} = \begin{array}{|c|c|c|} \hline \mathbf{L}' & \mathbf{F}'_1 & \mathbf{S}'_2 \\ \hline \mathbf{B}' & \mathbf{L} & \mathbf{S}_1 \\ \hline & \mathbf{B} & \mathbf{S}_0 \\ \hline \end{array}$$

where

- $\mathbf{S_k}$ describes the first transition from level ℓ ($\ell \geq 1$) to level $\ell + k$ without visiting levels $\ell + 1$ through $\ell + k - 1$.
- $\mathbf{S'_k}$ describes the first transition from level 0 to level k without visiting levels 1 through $k - 1$.

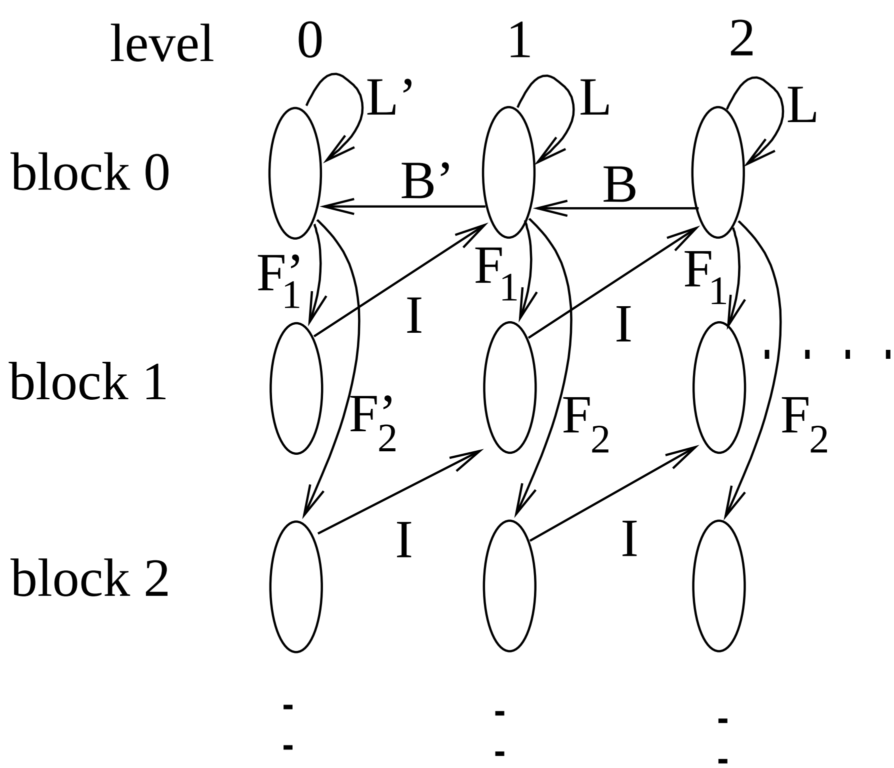
From $(\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2)\mathbf{Q}^{(0,1,2)} = 0$ we have

$$\boldsymbol{\pi}_2 = \left(\boldsymbol{\pi}_0 \mathbf{S}'_2 + \boldsymbol{\pi}_1 \mathbf{S}_1\right) (-\mathbf{S}_0)^{-1} \ .$$

Level by level increasing the size of the restricted process we obtain the Ramaswami formula.

Stationary solution of M/G/1-type process

We introduce an artificial infinite block structure of each levels to compose a QBD process.



Stationary solution of M/G/1-type process

Block structure of the infinite phase QBD process

$$
\mathbf{Q}' = \begin{array}{|c|c|c|c|c|}
\hline
\mathbb{L}' & \mathbb{F} & & & \\
\hline
\mathbb{B}' & \mathbb{L} & \mathbb{F} & & \\
\hline
 & \mathbb{B} & \mathbb{L} & \mathbb{F} & \\
\hline
 & & \mathbb{B} & \mathbb{L} & \mathbb{F} \\
\hline
 & & & \ddots & \ddots \\
\hline
\end{array}
\quad , \quad
\mathbb{F} = \begin{array}{|c|c|c|c|}
\hline
 & & & \\
\hline
\mathbf{I} & & & \\
\hline
 & \mathbf{I} & & \\
\hline
 & & \ddots & \\
\hline
\end{array}
\quad ,
$$

$$
\mathbb{L}' = \begin{array}{|c|c|c|c|}
\hline
\mathbf{L}' & \mathbf{F}'_1 & \mathbf{F}'_2 & \cdots \\
\hline
 & -\mathbf{I} & & \\
\hline
 & & -\mathbf{I} & \\
\hline
 & & & \ddots \\
\hline
\end{array}
\quad , \quad
\mathbb{B}' = \begin{array}{|c|c|c|c|}
\hline
\mathbf{B}' & & & \\
\hline
 & & & \\
\hline
 & & & \\
\hline
 & & & \\
\hline
\end{array}
\quad ,
$$

$$
\mathbb{L} = \begin{array}{|c|c|c|c|}
\hline
\mathbf{L} & \mathbf{F}_1 & \mathbf{F}_2 & \cdots \\
\hline
 & -\mathbf{I} & & \\
\hline
 & & -\mathbf{I} & \\
\hline
 & & & \ddots \\
\hline
\end{array}
\quad , \quad
\mathbb{B} = \begin{array}{|c|c|c|c|}
\hline
\mathbf{B} & & & \\
\hline
 & & & \\
\hline
 & & & \\
\hline
 & & & \\
\hline
\end{array}
\quad .
$$

Stationary solution of M/G/1-type process

At level 0 we have the stationary equation

$$0 = \pi_0' \mathbb{L}' + \pi_1' \mathbb{B}.$$

The partitioned form of this equation is

$$0 = \pi_{0,0}' \mathbf{L}' + \pi_{1,0}' \mathbf{B}', \quad \text{block 0,} \quad (0^*)$$

$$0 = -\pi_{0,i}' \mathbf{I} + \pi_{0,0}' \mathbf{F_i}', \quad \text{block i.} \quad (0^{**})$$

Stationary solution of M/G/1-type process

Form the transition structure of the QBD process we have

$$\mathbb{G} = \begin{array}{|c|c|c|c|}\hline \mathbf{G} & & & \\ \hline \mathbf{G}^2 & & & \\ \hline \mathbf{G}^3 & & & \\ \hline \vdots & & & \\ \hline \end{array} \, .$$

Restricting the QBD proces to the first n levels gives

$$\mathbf{Q_n'} = \begin{array}{|c|c|c|c|c|}\hline \mathbb{L}' & \mathbb{F} & & & \\ \hline \mathbb{B}' & \mathbb{L} & \ddots & & \\ \hline & \mathbb{B} & \ddots & \mathbb{F} & \\ \hline & & \ddots & \mathbb{L} & \mathbb{F} \\ \hline & & & \mathbb{B} & \mathbb{L}+\mathbb{F}\mathbb{G} \\ \hline \end{array} \, ,$$

Stationary solution of M/G/1-type process

where

$$\mathbb{F}\mathbb{G} = \begin{array}{|c|c|c|c|}\hline \mathbf{0} & & & \\ \hline \mathbf{G} & & & \\ \hline \mathbf{G}^2 & & & \\ \hline \vdots & & & \\ \hline \end{array},$$

and

$$\mathbb{L}+\mathbb{F}\mathbb{G} = \begin{array}{|c|c|c|c|}\hline \mathbf{L} & \mathbf{F_1} & \mathbf{F_2} & \cdots \\ \hline \mathbf{G} & -\mathbf{I} & & \\ \hline \mathbf{G}^2 & & -\mathbf{I} & \\ \hline \vdots & & & \ddots \\ \hline \end{array},$$

from which

$$0 = \pi'_{n-1}\mathbb{F} + \pi'_n(\mathbb{L}+\mathbb{F}\mathbb{G}).$$

The partitioned form of this equation is

$$0 = \pi'_{n-1,1} + \pi'_{n,0}\mathbf{L} + \sum_{k=1}^{\infty} \pi'_{n,k}\mathbf{G}^k, \quad \text{block 0,} \quad (*)$$

$$0 = \pi'_{n-1,i+1} - \pi'_{n,i} + \pi'_{n,0}\mathbf{F_i}, \qquad \text{block i.} \quad (**)$$

Stationary solution of M/G/1-type process

From (**) we have

$$\pi'_{n,i} = \pi'_{n-1,i+1} + \pi'_{n,0}\mathbf{F_i}.$$

Substituting $\pi'_{n,i}$ into (*) we have

$$\begin{aligned} 0 &= \pi'_{n-1,1} + \pi'_{n,0}\mathbf{L} + \sum_{k=1}^{\infty} \pi'_{n-1,k+1}\mathbf{G}^k + \sum_{k=1}^{\infty} \pi'_{n,0}\mathbf{F_k}\mathbf{G}^k, \\ &= \pi'_{n,0}\left(\mathbf{L} + \sum_{k=1}^{\infty} \mathbf{F_k}\mathbf{G}^k\right) + \sum_{k=0}^{\infty} \pi'_{n-1,k+1}\mathbf{G}^k, \end{aligned}$$

from which

$$\pi'_{n,0} = -\left(\sum_{k=0}^{\infty} \pi'_{n-1,k+1}\mathbf{G}^k\right) \underbrace{\left(\mathbf{L} + \sum_{i=1}^{\infty} \mathbf{F_i}\mathbf{G}^i\right)^{-1}}_{\mathbf{S_0}^{-1}}. \qquad (***)$$

Stationary solution of M/G/1-type process

Now, we look for a recursive evaluation of $\pi'_{n-1,i+1}$.

Applying (**) for block $i+1$ and level $n-1$ we have

$$\pi'_{n-1,i+1} = \pi'_{n-2,i+2} + \pi'_{n-1,0}\mathbf{F_{i+1}} \ ,$$

and similarly

$$\pi'_{n-2,i+2} = \pi'_{n-3,i+3} + \pi'_{n-2,0}\mathbf{F_{i+2}} \ ,$$

Repeatedly applying this up to level 0 we have:

$$\pi'_{n-1,i+1} = \pi'_{0,i+n} + \sum_{j=1}^{n-1} \pi'_{n-j,0}\mathbf{F_{i+j}} \ .$$

and using $\pi'_{0,i+j} = \pi'_{0,0}\mathbf{F'_{i+j}}$ from (0**) we have

$$\pi'_{n-1,i+1} = \pi'_{0,0}\mathbf{F'_{i+n}} + \sum_{j=1}^{n-1} \pi'_{n-j,0}\mathbf{F_{i+j}} \ .$$

Substituting this into (***) we have

$$\begin{aligned}
\pi'_{n,0} &= -\left(\sum_{i=0}^{\infty} \pi'_{n-1,i+1}\mathbf{G}^i\right)\mathbf{S_0}^{-1} = \\
&= -\left(\sum_{i=0}^{\infty} \pi'_{0,0}\mathbf{F}'_{\mathbf{i+n}}\mathbf{G}^i\right)\mathbf{S_0}^{-1} \\
&\quad -\left(\sum_{i=0}^{\infty}\sum_{j=1}^{n-1} \pi'_{n-j,0}\mathbf{F}_{\mathbf{i+j}}\mathbf{G}^i\right)\mathbf{S_0}^{-1} = \\
&= -\left(\pi'_{0,0}\mathbf{S}'_{\mathbf{n}}\right)\mathbf{S_0}^{-1} - \left(\sum_{j=1}^{n-1} \pi'_{n-j,0}\mathbf{S_j}\right)\mathbf{S_0}^{-1}
\end{aligned}$$

Finally, considering that the QBD process restricted to block 0 is equivalent with the M/G/1 type process we can establish the relation of their stationary probabilities:

$$\pi_n = \frac{\pi'_{n,0}}{\sum_{i=0}^{\infty} \pi'_{i,0}\mathbb{1}}.$$

Computation of π_0

Let $\mathbf{Q_0}$ be the generator of restricted CTMC of the original M/G/1-type process on level 0.

$$\mathbf{Q_0} = \mathbf{L}' + \sum_{k=1}^{\infty} \mathbf{F'_k P_{k\to 0}},$$

where

$$\mathbf{P_{k\to 0}} = Pr(J(\gamma_0) \mid X(0) = k, J(0)).$$

From the regular structure of the $k \geq 1$ levels we have $\mathbf{P_{k\to 0}} = \mathbf{G}^{k-1}\mathbf{P_{1\to 0}}$ and similar to the equation defining matrix $\mathbf{G}$ matrix $\mathbf{P_{1\to 0}}$ satisfies

$$\mathbf{B}' + \mathbf{L P_{1\to 0}} + \sum_{k=1}^{\infty} \mathbf{F_k G}^k \mathbf{P_{1\to 0}} = \mathbf{0},$$

from which

$$\mathbf{P_{1\to 0}} = -(\mathbf{L} + \sum_{k=1}^{\infty} \mathbf{F_k G}^k)^{-1}\mathbf{B}' = -\mathbf{S_0}^{-1}\mathbf{B}'$$

and

$$\mathbf{Q_0} = \mathbf{L}' + \sum_{k=1}^{\infty} \mathbf{F'_k G}^k(-\mathbf{S_0})^{-1}\mathbf{B}' = \mathbf{L}' + \mathbf{S'_1}(-\mathbf{S_0})^{-1}\mathbf{B}'.$$

π_0 satisfies $\pi_0 \mathbf{Q_0} = 0$.

Normalizing π_0

From

$$\boldsymbol{\pi}_i = \left(\boldsymbol{\pi}_0 \mathbf{S}'_{\mathbf{i}} + \sum_{k=1}^{i-1} \boldsymbol{\pi}_k \mathbf{S}_{\mathbf{i-k}}\right) (-\mathbf{S}_{\mathbf{0}})^{-1} \ , \quad \forall i \geq 1,$$

assuming $\mathbf{S}'_{\mathbf{0}} = \mathbf{0}$, $\widehat{\mathbf{S}}'(z) = \sum_{i=0}^{\infty} \mathbf{S}'_{\mathbf{i}} z^i$ and $\widehat{\mathbf{S}}(z) = \sum_{i=0}^{\infty} \mathbf{S}_{\mathbf{i}} z^i$ we have

$\widehat{\boldsymbol{\pi}}(z) = \sum_{i=0}^{\infty} \boldsymbol{\pi}_i z^i =$

$\boldsymbol{\pi}_0 + \boldsymbol{\pi}_0 \widehat{\mathbf{S}}'(z)(-\mathbf{S}_{\mathbf{0}})^{-1} + (\widehat{\boldsymbol{\pi}}(z) - \boldsymbol{\pi}_0)(\widehat{\mathbf{S}}(z) - \mathbf{S}_{\mathbf{0}})(-\mathbf{S}_{\mathbf{0}})^{-1}$

and than

$$\widehat{\boldsymbol{\pi}}(z) = \boldsymbol{\pi}_0 \left(\mathbf{I} - \widehat{\mathbf{S}}'(z)\widehat{\mathbf{S}}(z)^{-1}\right) \ .$$

The normalizing equation is

$$1 = \sum_{i=0}^{\infty} \boldsymbol{\pi}_i \mathbb{1} = \widehat{\boldsymbol{\pi}}(1)\mathbb{1} = \boldsymbol{\pi}_0 \mathbb{1} - \boldsymbol{\pi}_0 \widehat{\mathbf{S}}'(1)(\widehat{\mathbf{S}}(1))^{-1} \ \mathbb{1}$$

$$= \boldsymbol{\pi}_0 \mathbb{1} - \boldsymbol{\pi}_0 \left(\sum_{i=1}^{\infty} \mathbf{S}'_{\mathbf{i}}\right) \left(\sum_{j=0}^{\infty} \mathbf{S}_{\mathbf{j}}\right)^{-1} \mathbb{1}.$$

Normalizing π_0

Without introducing the transforms we have

$$\begin{aligned}
\sum_{i=1}^{\infty} \boldsymbol{\pi}_i &= \sum_{i=1}^{\infty} \boldsymbol{\pi}_0 \mathbf{S_i'}(-\mathbf{S_0})^{-1} + \sum_{i=1}^{\infty} \sum_{k=1}^{i-1} \boldsymbol{\pi}_k \mathbf{S_{i-k}}(-\mathbf{S_0})^{-1}, \\
&= \boldsymbol{\pi}_0 \sum_{i=1}^{\infty} \mathbf{S_i'}(-\mathbf{S_0})^{-1} + \left(\sum_{k=1}^{\infty} \boldsymbol{\pi}_k\right) \left(\sum_{i=1}^{\infty} \mathbf{S_i}\right) (-\mathbf{S_0})^{-1},
\end{aligned}$$

Multiplying with $-\mathbf{S_0}$ from the left gives

$$\sum_{i=1}^{\infty} \boldsymbol{\pi}_i \left(-\mathbf{S_0} - \sum_{i=1}^{\infty} \mathbf{S_i}\right) = \boldsymbol{\pi}_0 \sum_{i=1}^{\infty} \mathbf{S_i'}$$

from which we obtain the same normalizing equation

$$1 = \boldsymbol{\pi}_0 \mathbb{1} - \boldsymbol{\pi}_0 \left(\sum_{i=1}^{\infty} \mathbf{S_i'}\right) \left(\sum_{j=0}^{\infty} \mathbf{S_j}\right)^{-1} \mathbb{1}.$$

MAP/G/1 queue

(based on "Lucantoni: New results ..." paper)

Special case:
the M/G/1-type structure is resulted by a BMAP/G/1 queue with:

- BMAP arrival process: $\mathbf{D_0}, \mathbf{D_1}, \mathbf{D_2}, \ldots$
- (general) service time distribution: $H(t)$

Notations:

- number of arrivals in $(0, t)$: $N(t)$
- $\mathbf{D} = \sum_{i=0}^{\infty} \mathbf{D_i}$, $\mathbf{D}(z) = \sum_{i=0}^{\infty} \mathbf{D_i} z^i$
- arrival intensity: $\lambda = \boldsymbol{\gamma} \sum_{k=1}^{\infty} k \mathbf{D_k} \mathbb{1}$, where $\boldsymbol{\gamma}$ is the solution of $\boldsymbol{\gamma}\mathbf{D} = \mathbf{0}, \boldsymbol{\gamma}\mathbb{1} = 1$
- utilization: $\rho = \lambda/\mu$ ($1/\mu$ is the mean service time)
- $P_{ij}(n, t) = Pr(N(t) = n, J(t) = j \mid J(0) = i)$

$$\hat{\mathbf{P}}(z, t) = e^{\mathbf{D}(z)t}$$

MAP/G/1 queue

Stationary queue length at departure

Embedded DTMC:

$$\mathbf{P} = \begin{array}{|c|c|c|c|c|} \hline \mathbf{B_0} & \mathbf{B_1} & \mathbf{B_2} & \mathbf{B_3} & \ldots \\ \hline \mathbf{A_0} & \mathbf{A_1} & \mathbf{A_2} & \mathbf{A_3} & \ldots \\ \hline & \mathbf{A_0} & \mathbf{A_1} & \mathbf{A_2} & \ldots \\ \hline & & \mathbf{A_0} & \mathbf{A_1} & \ldots \\ \hline & & & \ddots & \ddots \\ \hline \end{array}$$

- $[\mathbf{A_n}]_{ij} =$
  Pr(phase moves from i to j and there are n arrivals during a service)

- $[\mathbf{B_n}]_{ij} =$
  Pr(phase moves from i to j and there are $n+1$ arrivals during an arrival and a service)

MAP/G/1 queue

$$\mathbf{A_n} = \int_{t=0}^{\infty} \mathbf{P}(n,t) dH(t), \qquad \mathbf{B_n} = -\mathbf{D_0}^{-1} \sum_{k=0}^{n} \mathbf{D_{k+1}} \mathbf{A_{n-k}}.$$

$$\begin{aligned} \mathbf{A(z)} &= \sum_{n=0}^{\infty} z^n \ \mathbf{A_n} = \sum_{n=0}^{\infty} z^n \int_{t=0}^{\infty} \mathbf{P}(n,t) dH(t) \\ &= \int_{t=0}^{\infty} \hat{\mathbf{P}}(z,t) dH(t) = \int_{t=0}^{\infty} e^{\mathbf{D(z)}t} dH(t) \end{aligned}$$

$$\mathbf{B(z)} = -\mathbf{D_0}^{-1} [\mathbf{D(z)} - \mathbf{D_0}] \ z^{-1} \mathbf{A(z)}.$$

MAP/G/1 queue

Stationary equation of the embedded process:

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_0 \mathbf{B_i} + \sum_{k=1}^{i+1} \boldsymbol{\pi}_k \mathbf{A_{i+1-k}}, \quad i \geq 0$$

Multiplying the ith equation with z^i and summing up gives:

$$\boldsymbol{\pi}(z) = \boldsymbol{\pi}_0 \mathbf{B}(z) + z^{-1}(\boldsymbol{\pi}(z) - \boldsymbol{\pi}_0)\mathbf{A}(z).$$

and the queue length distribution at departure is:

$$\begin{aligned} \boldsymbol{\pi}(z)\Big(z\mathbf{I} - \mathbf{A}(z)\Big) &= \boldsymbol{\pi}_0\Big(z\mathbf{B}(z) - \mathbf{A}(z)\Big) \\ &= \boldsymbol{\pi}_0(-\mathbf{D_0})^{-1}\mathbf{D}(z)\mathbf{A}(z), \end{aligned} \qquad (*)$$

Let $\widehat{\mathbf{G}}(z) = \sum_{n=0}^{\infty} z^n \mathbf{G}(n)$ where

$$G_{ij}(n) = Pr(J_{\gamma_0} = j, \gamma_0 = n \mid J_0 = i, N_0 = 1)$$

Transition from level i ($i \geq 1$) to level $i-1$:

$$\widehat{\mathbf{G}}(z) = z \sum_{k=0}^{\infty} \mathbf{A_k} \widehat{\mathbf{G}}^k(z), \quad \mathbf{G} = \sum_{k=0}^{\infty} \mathbf{A_k} \mathbf{G}^k.$$

Transition from level 0 to level 0:

$$\mathbf{K}(z) = z \sum_{k=0}^{\infty} \mathbf{B_k} \widehat{\mathbf{G}}^k(z), \quad \mathbf{K} = \sum_{k=0}^{\infty} \mathbf{B_k} \mathbf{G}^k.$$

MAP/G/1 queue

The unknown vector, $\boldsymbol{\pi}_0$, is calculated based on the stationary solution of the restricted process on level 0 ($\boldsymbol{\kappa}$) and the mean time to return to level 0 ($\boldsymbol{\kappa}^*$):

$$\boldsymbol{\pi}_0 = \frac{\boldsymbol{\kappa}}{\boldsymbol{\kappa}\boldsymbol{\kappa}^*}$$

where $\boldsymbol{\kappa}$ is the solution of $\boldsymbol{\kappa}\mathbf{K} = \boldsymbol{\kappa}$, $\boldsymbol{\kappa}\mathbb{1} = 1$, and the normalizing constant is computed from the mean time to return to level 0,

$$\boldsymbol{\kappa}^* = \frac{d}{dz}\mathbf{K}(z)\mathbb{1}\Big|_{z=1} \quad .$$

$\boldsymbol{\pi}_0$ can also be normalized based on $z \to 1$ in $(*)$.

MAP/G/1 queue

Computation of $\boldsymbol{\kappa}^*$

$$\boldsymbol{\kappa}^* = \frac{d}{dz}\mathbf{K}(z)\mathbb{1}\Big|_{z=1} = \frac{d}{dz} z \sum_{k=0}^{\infty} \mathbf{B_k}\widehat{\mathbf{G}}^k(z)\mathbb{1}\Big|_{z=1}$$

$$= \underbrace{\sum_{k=0}^{\infty} \mathbf{B_k} \underbrace{\mathbf{G}^k \mathbb{1}}_{\mathbb{1}}}_{\mathbb{1}} + \sum_{k=0}^{\infty} \mathbf{B_k} \frac{d}{dz}\widehat{\mathbf{G}}^k(z)\mathbb{1}|_{z=1}$$

$$= \underbrace{\sum_{k=0}^{\infty} \mathbf{B_k} \underbrace{\mathbf{G}^k \mathbb{1}}_{\mathbb{1}}}_{\mathbb{1}} + \sum_{k=0}^{\infty} \mathbf{B_k} \underbrace{\sum_{j=0}^{k-1} \mathbf{G}^j}_{(*)} \widehat{\mathbf{G}}(1) \underbrace{\mathbf{G}^{k-j-1}\mathbb{1}}_{\mathbb{1}}.$$

$(*)$ closed form expression is given at finite QBDs.

MAP/G/1 queue

Computation of the last term, $\widehat{\mathbf{G}}(1)\mathbb{1}$

$$\widehat{\mathbf{G}}(1)\mathbb{1} = \frac{d}{dz}\widehat{\mathbf{G}}(z)\mathbb{1}|_{z=1} = \frac{d}{dz} z \sum_{k=0}^{\infty} \mathbf{A_k} \widehat{\mathbf{G}}^k(z)\mathbb{1}\Big|_{z=1} =$$

$$= \underbrace{\sum_{k=0}^{\infty} \mathbf{A_k} \mathbf{G}^k \mathbb{1}}_{\mathbb{1}} + \sum_{k=1}^{\infty} \mathbf{A_k} \frac{d}{dz}\widehat{\mathbf{G}}^k(z)\mathbb{1}|_{z=1} =$$

$$= \mathbb{1} + \sum_{k=1}^{\infty} \mathbf{A_k} \sum_{j=0}^{k-1} \mathbf{G}^j \widehat{\mathbf{G}}(1) \mathbf{G}^{k-j-1} \mathbb{1} =$$

$$= \mathbb{1} + \sum_{k=1}^{\infty} \mathbf{A_k} \underbrace{\sum_{j=0}^{k-1} \mathbf{G}^j}_{(*)} \widehat{\mathbf{G}}(1)\mathbb{1} =$$

$(*)$ closed form expression.

MAP/G/1 queue

Stationary queue length distribution at arbitrary time:

The $(N(t), J(t))$ process of a MAP/G/1 queue is a Markov regenerative process with embedded points at departures.

The stationary distribution ($\boldsymbol{\psi}(z)$) can be computed based on the embedded distribution ($\boldsymbol{\pi}(z)$) and the mean time spent in different state in a regenerative period.

$$T_{ij}(k,\ell) = \\ E(\text{time in } (\ell, j) \text{ in a reg. period} \mid N(0) = k, J(0) = i)$$

For $k \leq \ell$, $k > 0$

$$\mathbf{T}(k,\ell) = \int_{t=0}^{\infty} \mathbf{P}(\ell - k, t) \; (1 - H(t)) \; dt \; .$$

For $\ell = k = 0$

$$\mathbf{T}(0,0) = \int_{t=0}^{\infty} e^{\mathbf{D_0} t} \; dt = (-\mathbf{D_0})^{-1} \; .$$

For $k = 0$, $\ell > 0$

$$\mathbf{T}(0,\ell) = \sum_{k=1}^{\ell} \underbrace{(-\mathbf{D_0})^{-1}\mathbf{D_k}}_{\text{1st arrival}} \int_{t=0}^{\infty} \mathbf{P}(\ell - k, t) \; (1 - H(t)) \; dt \; .$$

MAP/G/1 queue

From Markov regenerative theory

$$\psi_\ell = \frac{\displaystyle\sum_{k=0}^{\ell} \pi_k \mathbf{T}(k,\ell)}{\displaystyle\sum_{k=0}^{\infty} \pi_k \sum_{n=k}^{\infty} \mathbf{T}(k,n)\mathbb{1}} = \lambda \sum_{k=0}^{\ell} \pi_k \mathbf{T}(k,\ell) \ ,$$

where the denominator is the mean time of a regenerative period, i.e., mean inter-departure time. When the system is stable it equals to the mean inter-arrival time, $1/\lambda$.

For $\ell = 0$ we have

$$\boldsymbol{\psi}_0 = \lambda \boldsymbol{\pi}_0 (-\mathbf{D_0})^{-1}$$

which satisfies $\boldsymbol{\psi}_0 \mathbb{1} = 1 - \rho$, since $\boldsymbol{\pi}_0(-\mathbf{D_0})^{-1}\mathbb{1}$ is the mean idle time in a regeneration period.

MAP/G/1 queue

For $\ell > 0$ we multiply with z^ℓ and sum up from 1 to ∞

$$\psi(z) - \psi_0 =$$

$$\lambda\pi_0(-\mathbf{D_0})^{-1}(\mathbf{D(z)}-\mathbf{D_0}) \int_{t=0}^{\infty} \hat{\mathbf{P}}(z)\ (1-H(t))\ dt\ +$$

$$\lambda(\pi(z) - \pi_0) \int_{t=0}^{\infty} \hat{\mathbf{P}}(z)\ (1-H(t))\ dt =$$

$$\lambda\Big(\pi_0(-\mathbf{D_0})^{-1}\mathbf{D(z)} + \pi(z)\Big) \int_{t=0}^{\infty} \hat{\mathbf{P}}(z)\ (1-H(t))\ dt\ ,$$

where

$$\int_{t=0}^{\infty} \hat{\mathbf{P}}(z)(1 - H(t))\ dt =$$

$$\int_{t=0}^{\infty} e^{\mathbf{D(z)}t}\ dt - \int_{t=0}^{\infty} e^{\mathbf{D(z)}t} H(t)\ dt =$$

$$\int_{t=0}^{\infty} e^{\mathbf{D(z)}t}\ dt - \int_{t=0}^{\infty} (-\mathbf{D(z)})^{-1} e^{\mathbf{D(z)}t}\ dH(t) =$$

$$(-\mathbf{D(z)})^{-1}(\mathbf{I} - \mathbf{A(z)})\ .$$

Note that, $\mathbf{D(z)}$ and $\mathbf{A(z)}$ commutes.

MAP/G/1 queue

$$\psi(z) - \psi_0 =$$

$$\lambda\left(\pi_0(-\mathbf{D_0})^{-1}\mathbf{D(z)} + \pi(z)\right)(-\mathbf{D(z)})^{-1}(\mathbf{I} - \mathbf{A(z)}) =$$

$$\underbrace{\lambda\left(- \pi_0(-\mathbf{D_0})^{-1}\right.}_{-\psi_0} \left. + \pi(z)(-\mathbf{D(z)})^{-1}\right)(\mathbf{I} - \mathbf{A(z)}) =$$

$$- \psi_0 + \lambda\pi_0(-\mathbf{D_0})^{-1}\mathbf{A(z)} \ + \lambda\pi(z)(-\mathbf{D(z)})^{-1}(\mathbf{I} - \mathbf{A(z)})$$

Simplifying with ψ_0 and substituting $\boldsymbol{\pi}_0(-\mathbf{D_0})^{-1}\mathbf{D}(z)\mathbf{A}(z)$ according to $(*)$, using that $\mathbf{D(z)}$ and $\mathbf{A(z)}$ commutes, gives

$$\psi(z) = \lambda\pi(z)(z\mathbf{I} - \mathbf{A(z)})(\mathbf{D(z)})^{-1} +$$

$$\lambda\pi(z)(-\mathbf{D(z)})^{-1}(\mathbf{I} - \mathbf{A(z)}),$$

and we finally get

$$\boldsymbol{\psi}(z)\mathbf{D}(z) = \lambda(z-1)\boldsymbol{\pi}(z).$$

The inverse transformation gives

$$\psi_{\ell+1} = \left(\sum_{k=0}^{\ell} \psi_k \mathbf{D_{\ell+1-k}} - \lambda(\pi_\ell - \pi_{\ell+1})\right)(-\mathbf{D_0})^{-1} \ .$$

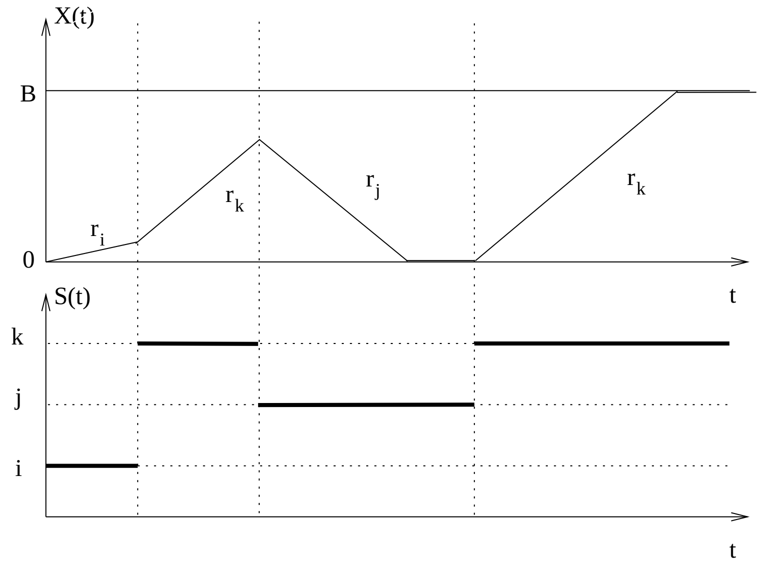
Fluid models

A simple function of the current state of a discrete state stochastic process, $S(t)$, governs the evolution of a continuous variable $X(t)$.

When the discrete state stochastic process is a CTMC

- $\{S(t), X(t)\}$ is a Markov process $\Rightarrow$ Markov fluid model.

Fluid models: bounded evolution of the continuous variable.



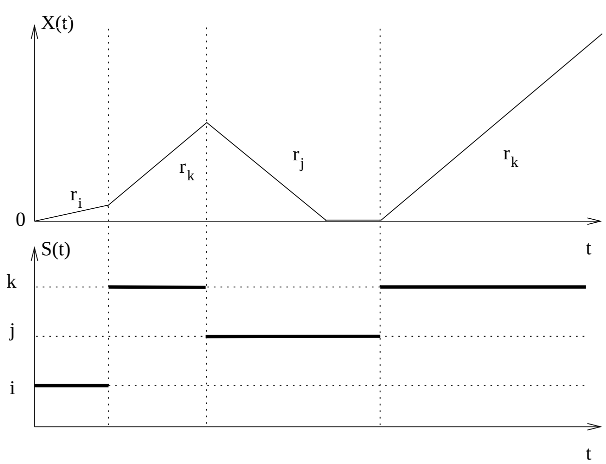
Classes of fluid models

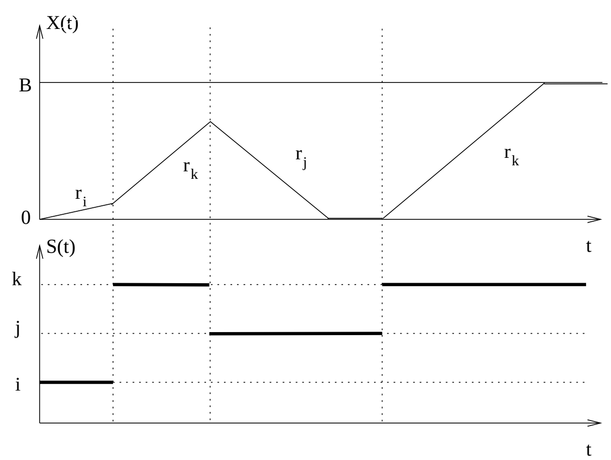
- finite buffer – infinite buffer,
- first order – second order,
- homogeneous – fluid level dependent,
- barrier behaviour in second order case
  - reflecting – absorbing.

Buffer size

Infinite buffer: $X(t)$ is only lower bounded at zero.

Finite buffer: $X(t)$ is lower bounded at zero and upper bounded at B.

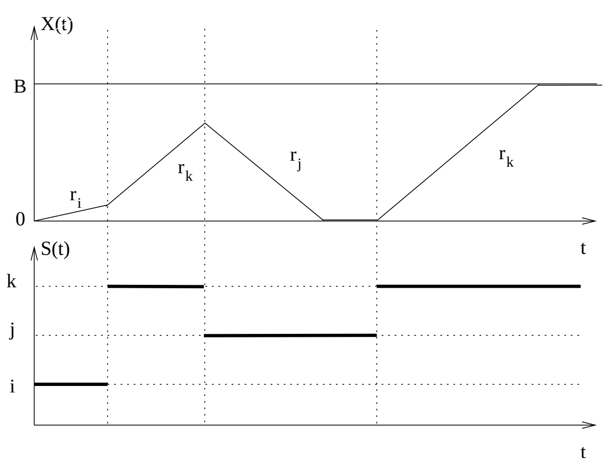


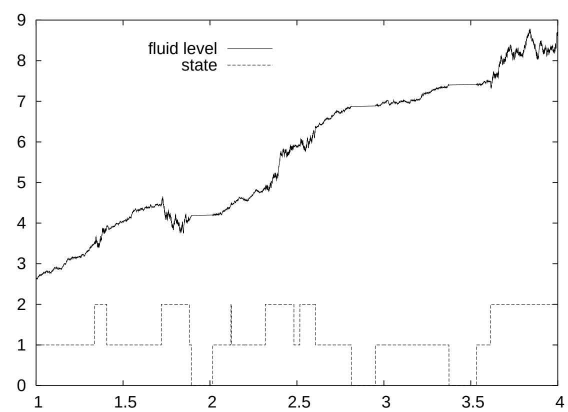


Fluid evolution

First order: the continuous quantity is a deterministic function of a CTMC.

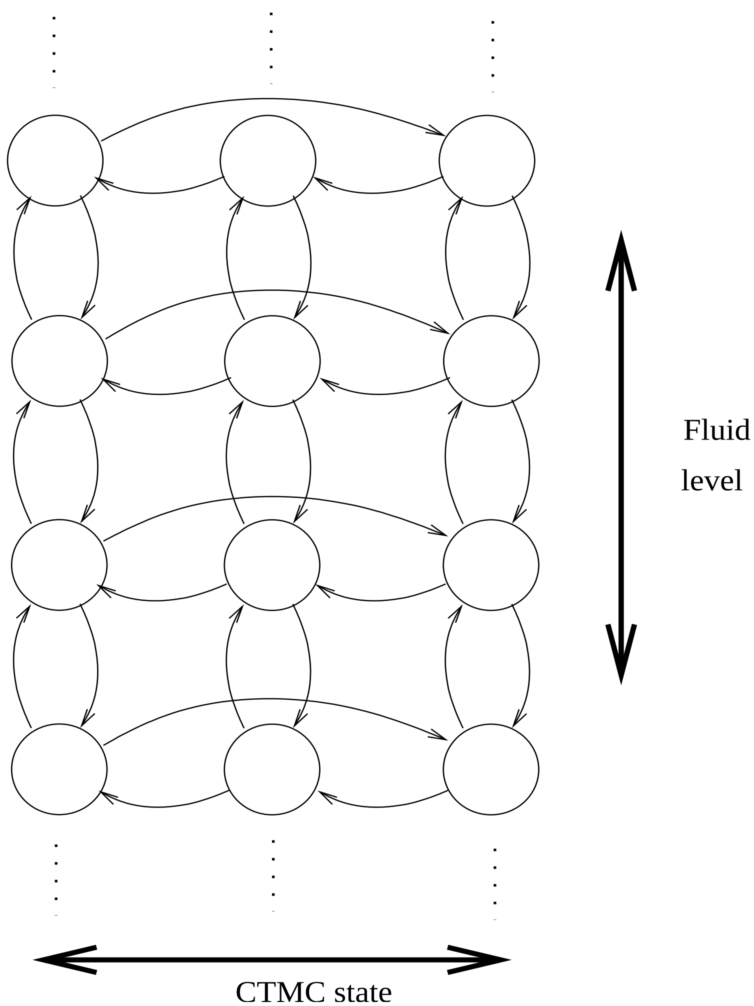
Second order: the continuous quantity is a stochastic function of a CTMC.





Interpretation of second order fluid models

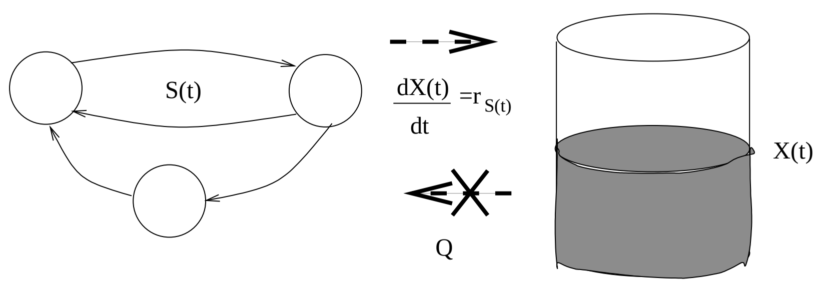
Random walk with decreasing time and fluid granularity.

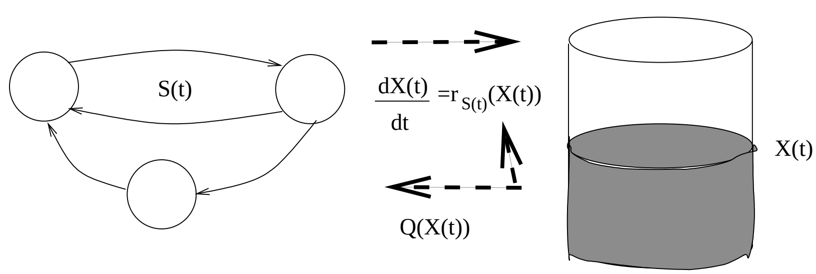


Dependence on fluid level

Homogeneous: the evolution of the CTMC is independent of the fluid level.

Fluid level dependent: the generator of the CTMC is a function of the fluid level.

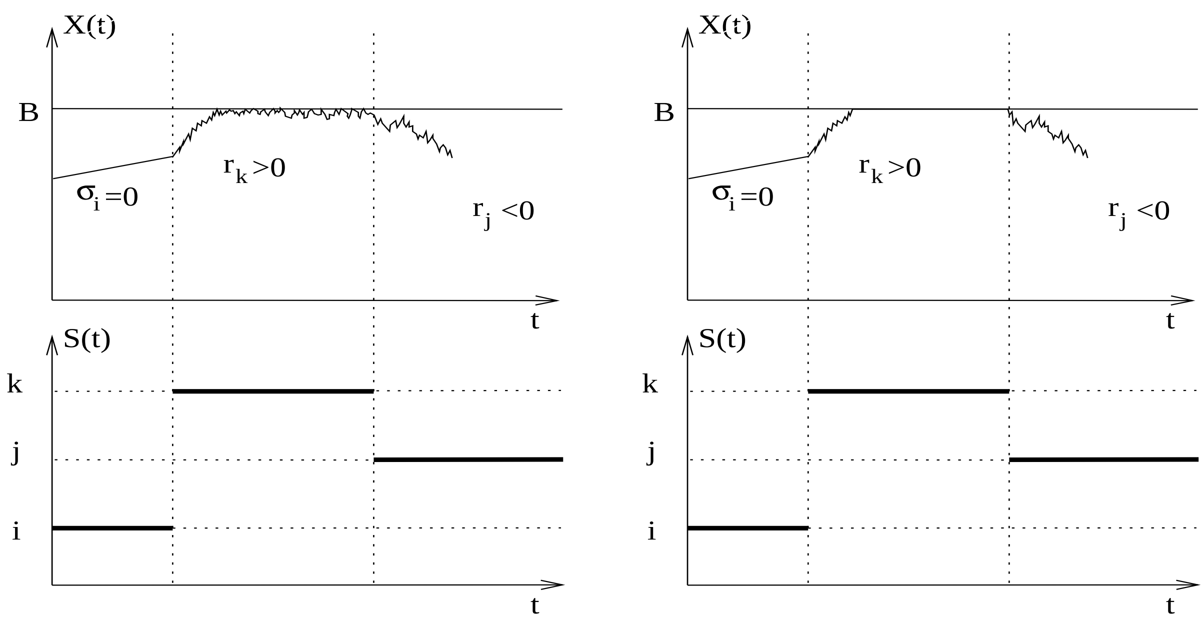




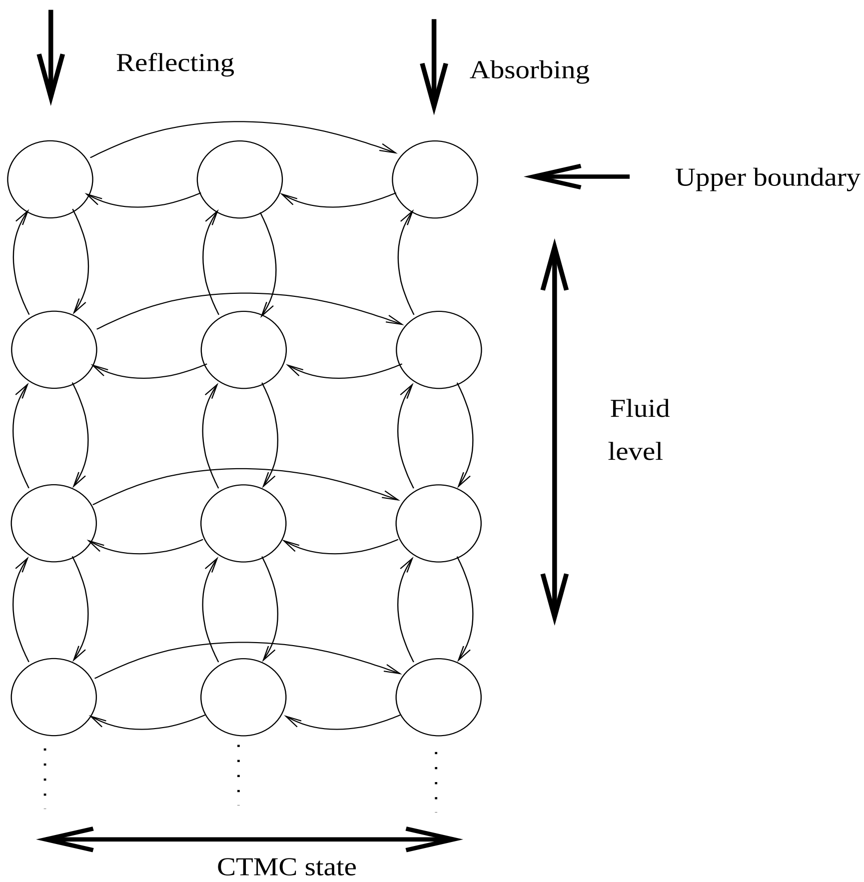
Boundary behaviour of second order fluid models

Reflecting: the fluid level is immediately reflected at the boundary.

Absorbing: the fluid level remains at the boundary up to a state transition of the Markov chain.



Interpretation of the boundary behaviours



Transient behaviour of fluid models

First order, infinite buffer, homogeneous case

During a sojourn of the CTMC in state i ($S(t) = i$) the fluid level ($X(t)$) increases at rate r_i when $X(t) > 0$:

$$X(t + \Delta) - X(t) = r_i\Delta \qquad \text{if } S(t) = i, X(t) > 0.$$

that is

$$\frac{d}{dt}X(t) = r_i \qquad \text{if } S(t) = i, X(t) > 0.$$

When $X(t) = 0$ the fluid level can not decrease:

$$\frac{d}{dt}X(t) = \max(r_i, 0) \qquad \text{if } S(t) = i, X(t) = 0.$$

That is

$$\frac{d}{dt}X(t) = \begin{cases} r_{S(t)} & \text{if } X(t) > 0, \\ \max(r_{S(t)}, 0) & \text{if } X(t) = 0. \end{cases}$$

Transient behaviour with finite buffer

When $X(t) = B$ the fluid level can not increase:

$$\frac{d}{dt}X(t) = \min(r_i, 0), \qquad \text{if } S(t) = i, X(t) = B.$$

That is

$$\frac{d}{dt}X(t) = \begin{cases} r_{S(t)}, & \text{if } X(t) > 0, \\ \max(r_{S(t)}, 0), & \text{if } X(t) = 0, \\ \min(r_{S(t)}, 0), & \text{if } X(t) = B. \end{cases}$$

3.2 Transient behaviour of fluid models

Second order, *infinite buffer*, *homogeneous* Markov fluid models with reflecting barrier

During a sojourn of the CTMC in state i ($S(t) = i$) in the sufficiently small $(t, t+\Delta)$ interval the distribution of the fluid increment $(X(t+\Delta) - X(t))$ is normal distributed with mean $r_i\Delta$ and variance $\sigma_i^2\Delta$:

$$X(t+\Delta) - X(t) = \mathcal{N}(r_i\Delta, \sigma_i^2\Delta),$$
$$\text{if } S(u) = i, u \in (t, t+\Delta), X(t) > 0.$$

At $X(t) = 0$ the fluid process is reflected immediately,

$\longrightarrow Pr(X(t) = 0) = 0.$

3.2 Transient behaviour of fluid models

Second order, infinite buffer, homogeneous Markov fluid models with absorbing barrier

Between the boundaries the evolution of the process is the same as before.

First time when the fluid level decreases to zero the fluid process stops,

$\longrightarrow Pr(X(t) = 0) > 0.$

Due to the absorbing property of the boundary the probability that the fluid level is close to it is very low,

$\longrightarrow \lim_{\Delta \to 0} \frac{Pr(0 < X(t) < \Delta)}{\Delta} = 0.$

3.2 Transient behaviour of fluid models

Inhomogeneous (fluid level dependent), first order, infinite buffer Markov fluid models

The evolution of the fluid level is the same:

$$\frac{d}{dt}X(t) = \begin{cases} r_{S(t)}(X(t)), & \text{if } X(t) > 0, \\ \max(r_{S(t)}(X(t)), 0), & \text{if } X(t) = 0. \end{cases}$$

But the evolution of the CTMC depends on the fluid level:

$$\lim_{\Delta \to 0} \frac{Pr(S(t+\Delta) = j | S(t) = i)}{\Delta} = q_{ij}(X(t)) \ .$$

The generator of the CTMC is $Q(X(t))$ and the rate matrix is $R(X(t))$.

3.3 Transient description of fluid models

Notations:

$\pi_i(t) = Pr(S(t) = i)$ – state probability,

$u_i(t) = Pr(X(t) = B, S(t) = i)$ – buffer full probability,

$\ell_i(t) = Pr(X(t) = 0, S(t) = i)$ – buffer empty probability,

$$p_i(t, x) = \lim_{\Delta \to 0} \frac{1}{\Delta} Pr(x < X(t) < x + \Delta, S(t) = i)$$

– fluid density.

$$\Longrightarrow \quad \pi_i(t) = \ell_i(t) + u_i(t) + \int_x p_i(t, x) dx.$$

3.3 Transient description of fluid models

First order, infinite buffer, homogeneous behaviour.

Forward argument:

If $S(t+\Delta) = i$, then between t and $t+\Delta$ the CTMC

- stays in i with probability $1 + q_{ii}\Delta$,
- moves from k to i with probability $q_{ki}\Delta$,
- has more than 1 state transition with probability $\sigma(\Delta)$.

3.3 Transient description of fluid models

Fluid density:

$$\begin{aligned} p_i(t+\Delta, x) &= (1+q_{ii}\Delta)\ p_i(t, x-r_i\Delta)+ \\ &\sum_{k\in\mathcal{S},k\neq i} q_{ki}\Delta\ p_k(t, x-\mathcal{O}(\Delta))+ \\ &\sigma(\Delta)\ , \end{aligned}$$

where $\lim_{\Delta\to 0}\sigma(\Delta)/\Delta = 0$ and $\lim_{\Delta\to 0}\mathcal{O}(\Delta) = 0$.

3.3 Transient description of fluid models

$$p_i(t+\Delta, x) - p_i(t, x - r_i\Delta) = \sum_{k\in\mathcal{S}} q_{ki}\Delta \; p_k(t, x - \mathcal{O}(\Delta)) + \sigma(\Delta) \ ,$$

$$\frac{p_i(t+\Delta, x) - p_i(t,x)}{\Delta} + r_i \frac{p_i(t,x) - p_i(t, x - r_i\Delta)}{r_i\Delta} = \sum_{k\in\mathcal{S}} q_{ki} \; p_k(t, x - \mathcal{O}(\Delta)) + \frac{\sigma(\Delta)}{\Delta} \ ,$$

$$\boxed{\frac{\partial}{\partial t} p_i(t,x) + r_i \frac{\partial}{\partial x} p_i(t,x) = \sum_{k\in\mathcal{S}} q_{ki} \; p_k(t,x) \ .}$$

3.3 Transient description of fluid models

Empty buffer probability:

If $r_i > 0$,

$\longrightarrow$ the fluid level increases in state i,

$\longrightarrow \ell_i(t) = Pr(X(t) = 0, S(t) = i) = 0.$

3.3 Transient description of fluid models

If $r_i \leq 0$:

$$
\begin{aligned}
\ell_i(t+\Delta) \ = \\
& (1+q_{ii}\Delta)\ \left(\ell_i(t) + \underbrace{\int_0^{-r_i\Delta} p_i(t,x)dx}_{*}\right) + \\
& \sum_{k\in\mathcal{S},k\neq i} q_{ki}\Delta\ \left(\ell_k(t) + \underbrace{\int_0^{\mathcal{O}(\Delta)} p_k(t,x)dx}_{\mathcal{O}(\Delta)}\right) + \\
& \sigma(\Delta)\ .
\end{aligned}
$$

3.3 Transient description of fluid models

When $x \leq -r_i\Delta$, then

$$p_i(t,x) = p_i(t,0) + xp_i'(t,0) + \sigma(\Delta) \ ,$$

and

$$\begin{aligned}
\boxed{*} &= \int_0^{-r_i\Delta} p_i(t,x)dx \\
&= \int_0^{-r_i\Delta} p_i(t,0)dx + \int_0^{-r_i\Delta} xp_i'(t,0)dx + \int_0^{-r_i\Delta} \sigma(\Delta)dx \\
&= -r_i\Delta \ p_i(t,0) \qquad + \underbrace{\frac{(-r_i\Delta)^2}{2} \ p_i'(t,0)}_{\sigma(\Delta)} \qquad + \underbrace{(-r_i\Delta) \ \sigma(\Delta)}_{\sigma(\Delta)} \ .
\end{aligned}$$

3.3 Transient description of fluid models

From which the empty buffer probability:

$$\ell_i(t+\Delta) \;=\; (1+q_{ii}\Delta)\,\Big(\ell_i(t) - r_i\Delta p_i(t,0) + \sigma(\Delta)\Big) + \sum_{k\in\mathcal{S},k\neq i} q_{ki}\Delta\;(\ell_k(t)+\mathcal{O}(\Delta)) + \sigma(\Delta)\;,$$

$$\ell_i(t+\Delta) - \ell_i(t) \;=\; q_{ii}\Delta\;\ell_i(t) - r_i\Delta p_i(t,0) + \sum_{k\in\mathcal{S},k\neq i} q_{ki}\Delta\;(\ell_k(t)+\mathcal{O}(\Delta)) + \sigma(\Delta)\;,$$

and

$$\frac{\ell_i(t+\Delta) - \ell_i(t)}{\Delta} \;=\; - r_i \; p_i(t,0) + \sum_{k \in \mathcal{S}} q_{ki} \;\; (\ell_k(t) + \mathcal{O}(\Delta)) + \frac{\sigma(\Delta)}{\Delta} \;,$$

$$\boxed{\frac{d}{dt}\ell_i(t) \;=\; -r_i \; p_i(t,0) + \sum_{k \in \mathcal{S}} q_{ki} \; \ell_k(t) \;.}$$

3.3 Transient description of fluid models

Set of governing equations:

Fluid density:

$$\frac{\partial}{\partial t} p_i(t,x) + r_i \frac{\partial}{\partial x} p_i(t,x) = \sum_{k \in \mathcal{S}} q_{ki}\ p_k(t,x)\ ,$$

Empty buffer probability:

if $r_i <= 0$:

$$\frac{d}{dt}\ell_i(t) \ = \ -r_i\ p_i(t,0) + \sum_{k \in \mathcal{S}} q_{ki}\ \ell_k(t),$$

if $r_i > 0$:

$$\ell_i(t) \ = 0.$$

3.3 Transient description of fluid models

By the definition of fluid density and empty buffer probability:

$$\int_0^\infty p_i(t,x)dx + \ell_i(t) = \pi_i(t) \ .$$

In the homogeneous case:

$$\frac{d}{dt}\pi_i(t) \ = \sum_{k\in\mathcal{S}} q_{ki} \ \pi_k(t), \quad \longrightarrow \quad \pi_i(t) = \pi_i(0)e^{Qt}.$$

3.3 Transient description of fluid models

First order, finite buffer, homogeneous behaviour.

If there is also an upper boundary:

if $r_i < 0$:

$$u_i(t) = 0,$$

if $r_i \geq 0$:

$$\frac{d}{dt}u_i(t) \;=\; r_i \; p_i(t,B) + \sum_{k\in\mathcal{S}} q_{ki} \; u_k(t).$$

3.3 Transient description of fluid models

Second order, infinite buffer, homogeneous behaviour.

Fluid density:

$$p_i(t+\Delta, x) = (1+q_{ii}\Delta) \underbrace{\int_{-\infty}^{\infty} p_i(t, x-u) f_{\mathcal{N}(\Delta r_i, \Delta\sigma_i^2)}(u) du}_{**} + \sum_{k\in\mathcal{S}, k\neq i} q_{ki}\Delta \; p_k(t, x-\mathcal{O}(\Delta)) + \sigma(\Delta)$$

3.3 Transient description of fluid models

Using

$$p_i(t,x-u) = p_i(t,x) - up_i'(t,x) + \frac{u^2}{2}p_i''(t,x) + \mathcal{O}(u)^3$$

we have:

$$\boxed{**} =$$

$$p_i(t,x)\underbrace{\int_{-\infty}^{\infty} f_{\mathcal{N}(\Delta r_i,\Delta\sigma_i^2)}(u)du}_{1} - p_i'(t,x)\underbrace{\int_{-\infty}^{\infty} u f_{\mathcal{N}(\Delta r_i,\Delta\sigma_i^2)}(u)du}_{\Delta r_i} +$$

$$p_i''(t,x)\underbrace{\int_{-\infty}^{\infty} \frac{u^2}{2} f_{\mathcal{N}(\Delta r_i,\Delta\sigma_i^2)}(u)du}_{\Delta^2 r_i^2+\Delta\sigma_i^2/2=\Delta\sigma_i^2/2+\sigma(\Delta)} + \underbrace{\int_{-\infty}^{\infty} \mathcal{O}(u)^3 f_{\mathcal{N}(\Delta r_i,\Delta\sigma_i^2)}(u)du}_{\mathcal{O}(\Delta)^2=\sigma(\Delta)}.$$

3.3 Transient description of fluid models

From which:

$$
\begin{aligned}
p_i(t+\Delta, x) \; = & \\
& (1+q_{ii}\Delta) \; \left(p_i(t,x) - p_i'(t,x)\Delta r_i + p_i''(t,x)\Delta \sigma_i^2/2\right) + \\
& \sum_{k\in\mathcal{S}, k\neq i} q_{ki}\Delta \; p_k(t, x-\mathcal{O}(\Delta)) + \quad \sigma(\Delta) \; ,
\end{aligned}
$$

$$
\begin{aligned}
p_i(t+\Delta, x) - p_i(t,x) \; = & \\
& q_{ii}\Delta p_i(t,x) - p_i'(t,x)\Delta r_i + p_i''(t,x)\Delta \sigma_i^2/2 + \\
& \sum_{k\in\mathcal{S}, k\neq i} q_{ki}\Delta \; p_k(t, x-\mathcal{O}(\Delta)) + \quad \sigma(\Delta) \; ,
\end{aligned}
$$

$$
\boxed{\frac{\partial}{\partial t} p_i(t,x) + \frac{\partial}{\partial x} p_i(t,x) r_i - \frac{\partial^2}{\partial x^2} p_i(t,x) \frac{\sigma_i^2}{2} \; = \sum_{k\in\mathcal{S}} q_{ki} \; p_k(t,x).}
$$

3.3 Transient description of fluid models

Second order, infinite buffer, reflecting barrier, homogeneous behaviour.

Boundary condition:

Reflecting barrier $\longrightarrow \ell_i(t) = 0$.

Fluid density at 0:

$$\int_0^\infty p_i(t,x)dx = \pi_i(t) \qquad \Big/ \ \frac{\partial}{\partial t}$$

3.3 Transient description of fluid models

$$\overbrace{\int_{x=0}^{\infty} \underbrace{\frac{\partial}{\partial t} p_i(t,x)} \, dx}^{-\frac{\partial p_i(t,x)}{\partial x} r_i + \frac{\partial^2 p_i(t,x)}{\partial x^2}\frac{\sigma_i^2}{2} + \sum_{k\in\mathcal{S}} q_{ki}\ p_k(t,x)} = \overbrace{\underbrace{\frac{\partial}{\partial t}\pi_i(t)}}^{\sum_{k\in\mathcal{S}} q_{ki}\pi_i(t)}$$

$$-r_i \underbrace{\Big[p_i(t,x)\Big]_{x=0}^{\infty}}_{-p_i(t,0)} + \frac{\sigma_i^2}{2} \underbrace{\Big[p_i'(t,x)\Big]_{x=0}^{\infty}}_{-p_i'(t,0)} + \sum_{k\in\mathcal{S}} q_{ki} \underbrace{\int_{x=0}^{\infty} p_k(t,x)dx}_{\pi_i(t)} = \sum_{k\in\mathcal{S}} q_{ki}\pi_i(t)$$

$$\boxed{r_i p_i(t,0) - \frac{\sigma_i^2}{2} p_i'(t,0) = 0}$$

3.3 Transient description of fluid models

First order, infinite buffer, inhomogeneous behaviour.

Fluid density:

$$\frac{\partial}{\partial t} p_i(t,x) + r_i(x) \frac{\partial}{\partial x} p_i(t,x) = \sum_{k \in \mathcal{S}} q_{ki}(x) \; p_k(t,x)$$

Empty buffer probability:

if $r_i(0) < 0$ (and $r_i(x)$ is continuous):

$$\frac{d}{dt}\ell_i(t) \;=\; -r_i(0) \; p_i(t,0) + \sum_{k \in \mathcal{S}} q_{ki}(0) \; \ell_k(t),$$

if $r_i(0) > 0$ (and $r_i(x)$ is continuous):

$$\ell_i(t) \;= 0.$$

3.3 Transient description of fluid models

General case:

Second order, *finite buffer*, *inhomogeneous behaviour*.

Differential equations:

$$\frac{\partial p(t,x)}{\partial t} + \frac{\partial p(t,x)}{\partial x} \boldsymbol{R}(x) - \frac{\partial^2 p(t,x)}{\partial x^2} \boldsymbol{S}(x) = p(t,x) \boldsymbol{Q}(x),$$

$$p(t,0) \boldsymbol{R}(0) - p'(t,0) \boldsymbol{S}(0) = \ell(t) \boldsymbol{Q}(0),$$

$$-p(t,B) \boldsymbol{R}(B) + p'(t,B) \boldsymbol{S}(B) = u(t) \boldsymbol{Q}(B),$$

where $\boldsymbol{R}(x) = \text{Diag}\langle r_i(x)\rangle$ and $\boldsymbol{S}(x) = \text{Diag}\langle \frac{\sigma_i^2(x)}{2}\rangle$.

3.3 Transient description of fluid models

General case:

Second order, *finite buffer*, *inhomogeneous behaviour*.

Bounding behaviour:

$\sigma_i = 0$ and positive/negative drift: $\ell_i(t) = 0 / u_i(t) = 0$.

$\sigma_i > 0$, reflecting lower/upper barrier: $\ell_i(t) = 0 / u_i(t) = 0$.

$\sigma_i > 0$, absor. lower/upper barrier: $p_i(t, 0) = 0 / p_i(t, B) = 0$.

Normalizing condition:

$$\int_0^B p(t,x)\ dx\,\mathbb{1} + \ell(t)\,\mathbb{1} + u(t)\,\mathbb{1} = 1.$$

3.4 Stationary description of fluid models

Condition of ergodicity:

For $\forall x, y \in \mathbb{R}^+, \forall i, j \in \mathcal{S}$ the transition time

$$T = \min_{t>0}(X(t) = y, S(t) = j | X(0) = x, S(0) = i)$$

has a finite mean (i.e., $E(T) < \infty$).

3.4 Stationary description of fluid models

Notations:

$\pi_i = \lim_{t\to\infty} Pr(S(t) = i)$ – state probability,

$u_i = \lim_{t\to\infty} Pr(X(t) = B, S(t) = i)$ – buffer full probability,

$\ell_i = \lim_{t\to\infty} Pr(X(t) = 0, S(t) = i)$ – buffer empty probability,

$$p_i(x) = \lim_{t\to\infty} \lim_{\Delta\to 0} \frac{1}{\Delta} Pr(x < X(t) < x + \Delta, S(t) = i)$$

– fluid density,

$$F_i(x) = \lim_{t\to\infty} Pr(X(t) < x, S(t) = i)$$

– fluid distribution.

3.4 Stationary description of fluid models

First order, infinite buffer, homogeneous behaviour.

Fluid density:

$$r_i \frac{\partial}{\partial x} p_i(x) = \sum_{k \in \mathcal{S}} q_{ki} \; p_k(x) \; .$$

Empty buffer probability:

if $r_i <= 0$:

$$0 \; = -r_i \; p_i(0) + \sum_{k \in \mathcal{S}} q_{ki} \; \ell_k,$$

if $r_i > 0$:

$$\ell_i \; = 0.$$

3.4 Stationary description of fluid models

First order, finite buffer, homogeneous behaviour.

Fluid density:

$$r_i \frac{\partial}{\partial x} p_i(x) = \sum_{k \in \mathcal{S}} q_{ki}\ p_k(x)\ .$$

Boundary equations:

$$\begin{cases} r_i\ p_i(0) = \sum_{k \in \mathcal{S}} q_{ki}\ \ell_k, & \text{if } r_i \le 0, \\ \ell_i\ = 0, & \text{if } r_i > 0. \end{cases}$$

$$\begin{cases} -r_i\ p_i(B) = \sum_{k \in \mathcal{S}} q_{ki}\ u_k, & \text{if } r_i \ge 0, \\ u_i\ = 0, & \text{if } r_i < 0. \end{cases}$$

3.4 Stationary description of fluid models

Second order, infinite buffer, reflecting boundary, homogeneous behaviour.

Fluid density:

$$r_i \frac{\partial}{\partial x} p_i(x) - \frac{\partial^2}{\partial x^2} p_i(x) \frac{\sigma_i^2}{2} = \sum_{k \in \mathcal{S}} q_{ki} \; p_k(x) \ .$$

Empty buffer probability:

$$\ell_i \; = 0.$$

Boundary equation:

$$r_i p_i(0) - \frac{\sigma_i^2}{2} p_i'(0) = \sum_{k \in \mathcal{S}} q_{ki} \; \ell_k = 0.$$

3.4 Stationary description of fluid models

Second order, infinite buffer, absorbing boundary, homogeneous behaviour.

Fluid density:

$$r_i \frac{\partial}{\partial x} p_i(x) - \frac{\partial^2}{\partial x^2} p_i(x) \frac{\sigma_i^2}{2} = \sum_{k \in \mathcal{S}} q_{ki} \; p_k(x).$$

Empty buffer probability:

$$p_i(0) = 0.$$

Boundary equation:

$$-\frac{\sigma_i^2}{2} p_i'(0) = \sum_{k \in \mathcal{S}} q_{ki} \; \ell_k.$$

3.4 Stationary description of fluid models

General case:

Second order, *finite buffer*, *inhomogeneous behaviour*.

$$
\begin{aligned}
p'(x)\,\boldsymbol{R}(x) - p''(x)\,\boldsymbol{S}(x) &= p(x)\,\boldsymbol{Q}(x)\,,\\
p(0)\,\boldsymbol{R}(0) - p'(0)\,\boldsymbol{S}(0) &= \ell\,\boldsymbol{Q}(0)\,,\\
-p(B)\,\boldsymbol{R}(B) + p'(B)\,\boldsymbol{S}(B) &= u\,\boldsymbol{Q}(B)\,,
\end{aligned}
$$

$\sigma_i = 0$ and positive/negative drift: $\ell_i = 0/u_i = 0$.

$\sigma_i > 0$, reflecting lower/upper barrier: $\ell_i = 0/u_i = 0$.

$\sigma_i > 0$, absorbing lower/upper barrier: $p_i(0) = 0/p_i(B) = 0$.

4 Solution methods

Numerical techniques:

	reward	fluid
differential equations	(+)	+
spectral decomposition	(+)	+
randomization	+	+
transform domain	+	+
matrix exponent	+	+
moments	+	-

4 Solution methods

Transient analysis:

- initial condition,
- set of differential equations,
- bounding behaviour.

Stationary analysis:

- set of differential equations,
- bounding behaviour,
- normalizing condition.

4.1 Transient solution methods

- Numerical solution of differential equations,
- Randomization,
- Markov regenerative approach,
- Transform domain.

4.1 Transient solution methods

Numerical solution of differential equations (Chen et al.)

All cases.

The approach

- starts from the initial condition, and
- follows the evolution of the fluid distribution in the $(t, t + \Delta)$ interval at some fluid levels based on the differential equations and the boundary condition.

This is the only approach for inhomogeneous models.

4.1 Transient solution methods

Randomization (Sericola)

First order, infinite buffer, homogeneous behaviour.

$$F_i^c(t,x) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\sum_{k=0}^{n}\binom{n}{k} x_j^k (1-x_j)^{n-k} b_i^{(j)}(n,k),$$

where $F_i^c(t,x) = Pr(X(t) > x, S(t) = i)$,

$x_j = \frac{x - r_{j-1}^+ t}{r_j t - r_{j-1}^+ t}$ if $x \in [r_{j-1}^+ t, r_j t)$, and

$b_i^{(j)}(n,k)$ is defined by initial value and a simple recursion.

4.1 Transient solution methods

Properties of the randomization based solution method:

- the expression with the given recursive formulas is a solution of the differential equation,
  the initial value of $b_i^{(j)}(n,k)$ is set to fulfill the boundary condition,

- $0 \leq x_j \leq 1$
  $\longrightarrow$ convex combination of non-negative numbers
  $\longrightarrow$ numerical stability,

- the initial fluid level is $X(0) = 0$.
  (extension to $X(0) > 0$ and to finite buffer is not available.)

4.1 Transient solution methods *First order, infinite buffer, homogeneous case.*

Markov regenerative approach (Ahn-Ramaswami)

Busy/idle period:
interval when the buffer is non-empty/empty.

T_i : the beginning of the ith busy period.

$\Longrightarrow (S(t_i), T_i)$ is a Markov renewal sequence.

The idle period is PH distributed.

Analysis of a single busy period:

similar analysis as in Matrix geometric models.

4.1 Transient solution methods

First order, infinite/finite buffer, homogeneous case.

Transform domain description (Ren-Kobayashi)

The Laplace transform of

$$\frac{\partial p(t,x)}{\partial t} + \frac{\partial p(t,x)}{\partial x}\,\boldsymbol{R} - \frac{\partial^2 p(t,x)}{\partial x^2}\,\boldsymbol{S} \;= p(t,x)\,\boldsymbol{Q}\;,$$

is

$$p^{**}(s,v) = (\quad \underbrace{p^*(0,v)}_{\text{initial condition}} \quad + \underbrace{p^*(s,0)}_{\text{unknown}}\;\boldsymbol{R})(s\boldsymbol{I} + v\boldsymbol{R} - \boldsymbol{Q})^{-1}.$$

$p^*(s,0)$ eliminates the roots of $\det(s\boldsymbol{I} + v\boldsymbol{R} - \boldsymbol{Q})$.

4.2 Stationary solution methods

Condition of stability of infinite buffer first/second order homogeneous fluid models.

Suppose $S(t)$ is a finite state irreducible CTMC with stationary distribution π.

The fluid model is stable if the overall drift is negative:

$$d = \sum_{i \in \mathcal{S}} \pi_i r_i < 0.$$

$\longrightarrow$ the variance does not play role.

4.2 Stationary solution methods

- Spectral decomposition,
- Matrix exponent,
- Numerical solution of differential equations,
- Randomization.

4.2 Stationary solution methods

State space partitioning:

- $\mathcal{S}^\sigma$: $i \in \mathcal{S}^\sigma$ iff $\sigma_i > 0$,
  second order states,

- $\mathcal{S}^0$: $i \in \mathcal{S}^0$ iff $r_i = 0$ and $\sigma_i = 0$,
  zero states,

- $\mathcal{S}^+$: $i \in \mathcal{S}^+$ iff $r_i > 0$ and $\sigma_i = 0$,
  positive first order states,

- $\mathcal{S}^-$: $i \in \mathcal{S}^-$ iff $r_i < 0$ and $\sigma_i = 0$,
  negative first order states,

- $\mathcal{S}^\pm = \mathcal{S}^- \bigcup \mathcal{S}^+$,
  first order states.

4.2 Stationary solution methods

First order, infinite/finite buffer, homogeneous case.

Spectral decomposition (Kulkarni)

Differential equation: $p'(x)\,\boldsymbol{R} = p(x)\,\boldsymbol{Q}$,

Form of the solution vector: $p(x) = e^{\lambda x}\phi$,

Substituting this solution we get the characteristic equation:

$$\phi(\lambda\boldsymbol{R} - \boldsymbol{Q}) = 0,$$

whose solutions are obtained at $\det(\lambda\boldsymbol{R} - \boldsymbol{Q}) = 0$.

4.2 Stationary solution methods

Spectral decomposition

The characteristic equation of a stable model has $|\mathcal{S}^{\pm}| = |\mathcal{S}^{+}| + |\mathcal{S}^{-}|$ solutions, with

$$\begin{cases} |\mathcal{S}^{+}| & \text{negative eigenvalue,} \\ 1 & \text{zero eigenvalue,} \\ |\mathcal{S}^{-}| - 1 & \text{positive eigenvalue.} \end{cases}$$

From which the solution is: $$p(x) = \sum_{j=1}^{|\mathcal{S}^{\pm}|} a_j e^{\lambda_j x} \phi_j,$$

and the a_j coefficients are set to fulfill the boundary and normalizing conditions.

4.2 Stationary solution methods

Spectral decomposition

In the *infinite buffer* case these conditions are:

- $p(0)\,\boldsymbol{R} = \ell\,\boldsymbol{Q}$,
- $\ell_i = 0$ if $r_i > 0$, and
- $\int_0^\infty p_i(x) + \ell_i = \pi_i$.

From which $a_j = 0$ for $\lambda_j > 0$
and the rest of the coefficients are obtained from a linear system of equations.

4.2 Stationary solution methods

Spectral decomposition

In the *finite buffer* case these conditions are:

- $p(0)\,\boldsymbol{R} = \ell\,\boldsymbol{Q}\,, \quad p(B)\,\boldsymbol{R} = u\,\boldsymbol{Q}\,,$
- $\ell_i = 0$ if $r_i > 0$, $u_i = 0$ if $r_i < 0$, and
- $\int_0^\infty p_i(x) + \ell_i + u_i = \pi_i$.

From which the a_j coefficients are obtained from a linear system of equations.

4.2 Stationary solution methods

Consequences:

- If $|\mathcal{S}^-| = 1$
  $\longrightarrow$ all eigenvalues are non-positive.

- If $|\mathcal{S}^-| > 1$ and the buffer is infinite
  $\longrightarrow$ special treatment of the positive eigenvalues
  $\longrightarrow$ spectral decomposition is necessary.

- If the buffer is finite
  $\longrightarrow$ no need for special treatment of the positive eigenvalues.

4.2 Stationary solution methods

First order, finite buffer, homogeneous case.

Matrix exponent: (Gribaudo)

Assume that $|\mathcal{S}^0| = 0$ and $\mathcal{S} = \mathcal{S}^{\pm}$.

Introduce $v = \ell + u$, $\boldsymbol{Q}^-$, $\boldsymbol{Q}^+$,

where $q^-_{ij} = q_{ij}$ if $i \in \mathcal{S}^-$ and otherwise $q^-_{ij} = 0$.

The set of equations becomes:

$$\frac{\partial p(x)}{\partial x}\boldsymbol{R} = p(x)\boldsymbol{Q} \quad \longrightarrow \quad p(B) = p(0)\ e^{\boldsymbol{Q}\boldsymbol{R}^{-1}B} = p(0)\ \boldsymbol{\Phi},$$

$$p(0)\boldsymbol{R} = v\boldsymbol{Q}^- \quad \longrightarrow \quad p(0) = v\boldsymbol{Q}^-\boldsymbol{R}^{-1},$$

$$-p(B)\boldsymbol{R} = v\boldsymbol{Q}^+ \quad \longrightarrow \quad \boxed{v(\boldsymbol{Q}^-\boldsymbol{R}^{-1}\boldsymbol{\Phi}\boldsymbol{R} + \boldsymbol{Q}^+) = 0},$$

4.2 Stationary solution methods

Matrix exponent:

And the normalizing condition is

$$\ell \mathbb{1} + u \mathbb{1} + p(0) \underbrace{\int_0^B e^{\boldsymbol{Q}\boldsymbol{R}^{-1}x} dx}_{\boldsymbol{\Psi}} \mathbb{1} =$$

$$\boxed{v(\boldsymbol{I} + \boldsymbol{Q}^{-}\boldsymbol{R}^{-1}\boldsymbol{\Psi})\mathbb{1} = 1}.$$

4.2 Stationary solution methods

Relation of spectral decomposition and matrix exponent:

Assume that $|\mathcal{S}^0| = 0$ and $\mathcal{S} = \mathcal{S}^{\pm}$.

The characteristic equation is: $\phi(\lambda \boldsymbol{I} - \boldsymbol{Q}\boldsymbol{R}^{-1}) = 0$,

The spectral solution is: $$p(x) = \sum_{j=1}^{|\mathcal{S}|} a_j e^{\lambda_j x} \phi_j,$$

where λ_j and ϕ_j are the eigenvalues and the left eigen-vector of matrix $\boldsymbol{Q}\boldsymbol{R}^{-1}$.

4.2 Stationary solution methods

Relation of spectral decomposition and matrix exponent:

Introducing $a = \{a_j\}$ and $\boldsymbol{B} = \begin{pmatrix} \hline \phi_1 \\ \hline \phi_2 \\ \hline \vdots \\ \hline \phi_{|\mathcal{S}^{\pm}|} \\ \hline \end{pmatrix}$,

the spectral solution can be rewritten as:

$$\begin{aligned} p(x) &= \sum_{j=1}^{|\mathcal{S}|} a_j e^{\lambda_j x} \phi_j = a \ \text{Diag}\langle e^{\lambda_i x} \rangle \ \boldsymbol{B} \\ &= \underbrace{a \ \boldsymbol{B}}_{= p(0)} \ \underbrace{\boldsymbol{B}^{-1} \ \text{Diag}\langle e^{\lambda_i x} \rangle \ \boldsymbol{B}}_{e^{\boldsymbol{Q}\boldsymbol{R}^{-1} x}}, \end{aligned}$$

4.2 Stationary solution methods

Second order, infinite/finite buffer, homogeneous case.

Spectral decomposition (Karandikar-Kulkarni)

Differential equation: $p'(x)\,\boldsymbol{R} - p''(x)\,\boldsymbol{S} = p(x)\,\boldsymbol{Q}$,

Form of the solution vector: $p(x) = e^{\lambda x}\phi$,

Substituting this solution we get the characteristic equation:

$$\phi(\lambda \boldsymbol{R} - \lambda^2 \boldsymbol{S} - \boldsymbol{Q}) = 0,$$

whose solutions are obtained at $\det(\lambda \boldsymbol{R} - \lambda^2 \boldsymbol{S} - \boldsymbol{Q}) = 0$.

4.2 Stationary solution methods

Spectral decomposition

The characteristic equation of a stable model has $2|\mathcal{S}^\sigma|+|\mathcal{S}^\pm|$ solutions, with

$$\begin{cases} |\mathcal{S}^\sigma|+|\mathcal{S}^+| & \text{negative eigenvalue,} \\ 1 & \text{zero eigenvalue,} \\ |\mathcal{S}^\sigma|+|\mathcal{S}^-|-1 & \text{positive eigenvalue.} \end{cases}$$

From which the solution is: $\quad p(x) = \sum_{j=1}^{2|\mathcal{S}^\sigma|+|\mathcal{S}^\pm|} a_j e^{\lambda_j x}\phi_j,$

and the a_j coefficients are set to fulfill the boundary and normalizing conditions.

4.2 Stationary solution methods

Second order, infinite/infinite buffer, homogeneous case.

A transformation of the quadratic equation to a linear one

Assume that $|\mathcal{S}^0| = |\mathcal{S}^\pm| = 0$ and $\mathcal{S} = \mathcal{S}^\sigma$.

$$\begin{aligned} \frac{d}{dx}p(x)\,\boldsymbol{R} - \frac{d}{dx}p'(x)\,\boldsymbol{S} &= p(x)\,\boldsymbol{Q}\,, \\ \frac{d}{dx}p(x)\,\boldsymbol{I} &= p'(x)\,\boldsymbol{I}\,, \end{aligned}$$

$$\frac{d}{dx}\begin{array}{|c|c|}\hline p(x) & p'(x) \\ \hline \end{array}\ \begin{array}{|c|c|}\hline \boldsymbol{R} & \boldsymbol{I} \\ \hline -\boldsymbol{S} & \boldsymbol{0} \\ \hline \end{array} = \begin{array}{|c|c|}\hline p(x) & p'(x) \\ \hline \end{array}\ \begin{array}{|c|c|}\hline \boldsymbol{Q} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{I} \\ \hline \end{array}$$

$$\Longrightarrow \quad \frac{d}{dx}\widehat{p}(x)\,\hat{\boldsymbol{R}} = \widehat{p}(x)\,\hat{\boldsymbol{Q}} \quad \longrightarrow \quad \widehat{p}(B) = \widehat{p}(0)\ e^{\hat{\boldsymbol{Q}}\hat{\boldsymbol{R}}^{-1}B}.$$

4.2 Stationary solution methods

Numerical solution of differential equations (Gribaudo et al.)

All cases with finite buffer.

Numerically solve the matrix function $\boldsymbol{M}(x)$ with initial condition $\boldsymbol{M}(0) = \boldsymbol{I}$ based on

$$\boldsymbol{M}'(x)\,\boldsymbol{R}(x) - \boldsymbol{M}''(x)\,\boldsymbol{S}(x) = \boldsymbol{M}(x)\,\boldsymbol{Q}(x)$$

and calculate the unknown boundary conditions based on

$$p(B) = p(0)\ \boldsymbol{M}(B)$$

This is the only approach for inhomogeneous models.

4.2 Stationary solution methods

First order, infinite/finite buffer, homogeneous case.

Randomization (Sericola)

Randomization with simple coefficients:

$$F_i(x) = \sum_{n=0}^{\infty} e^{-\lambda t/r} \frac{(\lambda t/r)^n}{n!} \ b_i(n)$$

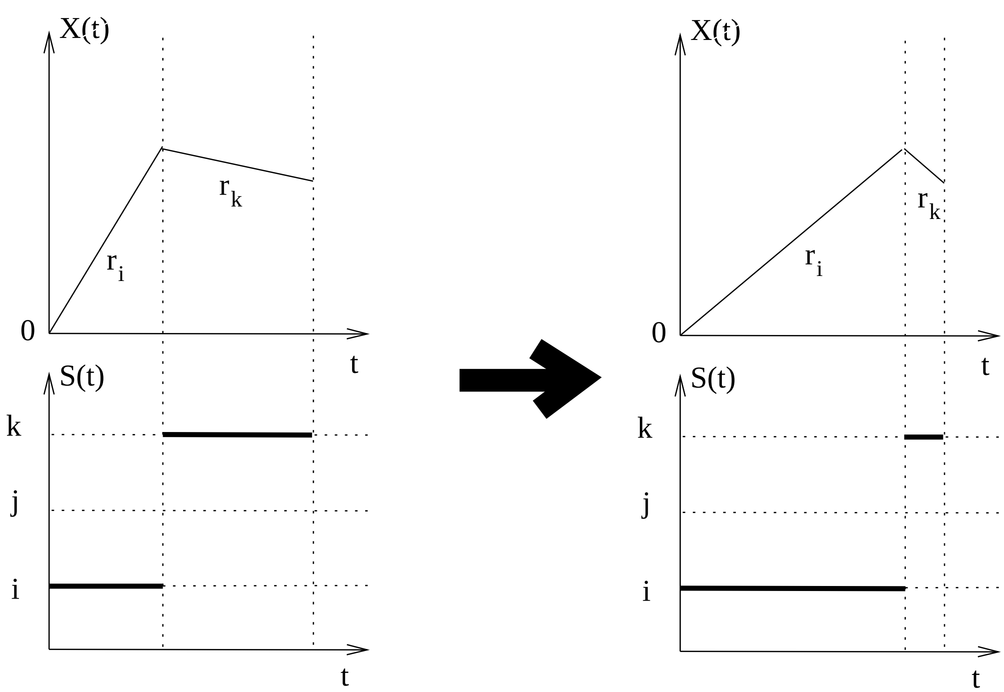
where $r = \min(r_i | r_i > 0)$ and
$b_i(n)$ is defined by initial value and a simple recursion.

Applicable only when $|\mathcal{S}^-| = 1$.

Matrix analytic solution for infinite buffer

A model transformation is proposed by Soares and Latouche:

$$\{\boldsymbol{Q}, \boldsymbol{R}\} \quad \rightarrow \quad \left\{\hat{\boldsymbol{Q}} = \boldsymbol{Q} \cdot \text{diag}\langle \frac{1}{|r_i|} \rangle, \hat{\boldsymbol{R}} = \begin{array}{|c|c|} \hline \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{I} \\ \hline \end{array} \right\}$$



Matrix analytic solution

The partitioned form of the differential equation and the boundary conditions are

$$\frac{d}{dx}[p_+(x)|p_-(x)]\left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & -\mathbf{I} \end{array}\right] = [p_+(x)|p_-(x)]\left[\begin{array}{c|c} \mathbf{Q}_{++} & \mathbf{Q}_{+-} \\ \hline \mathbf{Q}_{-+} & \mathbf{Q}_{--} \end{array}\right].$$

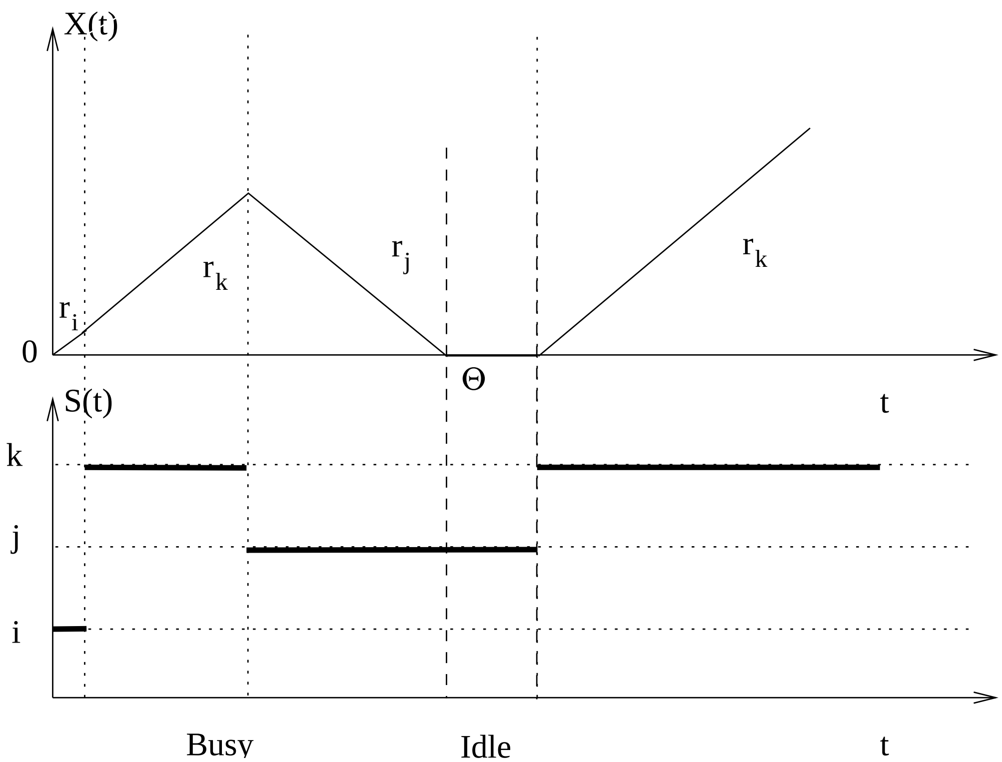
$$\ell_+(0) = \mathbf{0},$$

$$p_-(0) + \boldsymbol{\ell}_-(0)\mathbf{Q}_{--} = \mathbf{0},$$

and

$$p_+(0) = \ell_-(0)\mathbf{Q}_{-+}.$$

Busy-idle periods of the buffer



Idle period:

We have $S(t) \in \mathcal{S}^-$ while $X(t) = 0$.

Length of the idle period: $\Omega = \sup(t : X(t) = 0)$,

PH distributed.

State transition during the idle period:

$$Pr(S(\Omega) = j \in \mathcal{S}^+ | S(0) = i \in \mathcal{S}^-, X(0) = 0)$$
$$= [(-\boldsymbol{Q}^{--})^{-1}\boldsymbol{Q}^{-+}]_{ij}.$$

Analysis of the busy period

Busy period:

Length of the busy period: $\Theta = \min(t : X(t) = 0)$

State transition during the busy period:

$$\Psi_{ij} = Pr(S(\Theta) = j \in \mathcal{S}^- | S(0) = i \in \mathcal{S}^+, X(0) = 0).$$

Analysis of the busy period

Theorem:

$$p(x) = \ell^{-}\boldsymbol{Q}^{-+}\boldsymbol{N}(x),$$

where

$N_{ij}(x) =$

$$E(\#t^* : t^* < \Theta, X(t^*) = x, S(t^*) = j | X(0) = 0, S(0) = i)$$

is the mean number of level crossings at level x in state j during a busy period.

Proof:

$$\begin{aligned} P_j(t,x) = & \sum_{i \in \mathcal{S}^-} \sum_{k \in \mathcal{S}^+} \int_{\tau=0}^{t} P_i(t-\tau, 0)[\boldsymbol{Q}^{-+}]_{ik} P^0_{kj}(x,\tau) d\tau \\ & + \sum_{i \in \mathcal{S}^+} Pr(S(0) = i) P^0_{ij}(x,t) \end{aligned}$$

where $P_j(t,x) = Pr(X(t) < x, S(t) = j)$ and

$P^0_{ij}(t,x) = Pr(X(t) < x, S(t) = j, t < \Theta | X(0) = 0, S(0) = i)$.

Analysis of the busy period

Proof (cont.):

The derivative of $P_j(t,x)$ with respect to x is

$$\begin{aligned}\frac{\partial}{\partial x}P_j(t,x) = &\sum_{i\in\mathcal{S}^-}\sum_{k\in\mathcal{S}^+}\int_{\tau=0}^{t}\ell_i(t-\tau,0)[\boldsymbol{Q}^{-+}]_{ik}\frac{\partial}{\partial x}P^0_{kj}(x,\tau)d\tau \\ &+\sum_{i\in\mathcal{S}^+}\ell_i(0)\frac{\partial}{\partial x}P^0_{ij}(x,t)\ .\end{aligned}$$

As $t\to\infty$ it gets

$$p_j(x) = \sum_{i\in\mathcal{S}^-}\sum_{k\in\mathcal{S}^+}\ell_i[\boldsymbol{Q}^{-+}]_{ik}\int_{\tau=0}^{\infty}\frac{\partial}{\partial x}P^0_{kj}(x,\tau)d\tau\ ,$$

since $P^0_{ij}(t,x)\to 0$ as $t\to\infty$.

$$\begin{aligned}&\int_{\tau=0}^{\infty}\frac{\partial}{\partial x}P^0_{kj}(x,\tau)d\tau = \\ &\int_{\tau=0}^{\infty}\lim_{\Delta\to 0}\frac{P^0_{kj}(x+\Delta,\tau)-P^0_{kj}(x,\tau)}{\Delta}d\tau = \\ &\int_{\tau=0}^{\infty}\lim_{\Delta\to 0}\frac{Pr(x\le X(\tau)<x+\Delta, S(\tau)=j, \tau<\Theta|X(0)=0, S(0)=k)}{\Delta}d\tau = \\ &\int_{\tau=0}^{\infty}\lim_{\Delta\to 0}\frac{E(I_{\{x\le X(\tau)<x+\Delta, S(\tau)=j,\tau<\Theta|X(0)=0,S(0)=k\}})}{\Delta}d\tau = \\ &N_{kj}(x)\ ,\end{aligned}$$

Analysis of the busy period

Theorem:

$$\boldsymbol{N}(x) = e^{\boldsymbol{K}x}[\boldsymbol{I} \quad \boldsymbol{\Psi}],$$

Proof:

Due to level independency

$$\boldsymbol{N}^{++}(x+y) = \boldsymbol{N}^{++}(x)\boldsymbol{N}^{++}(y),$$

consequently

$$\boldsymbol{N}^{++}(x) = e^{\boldsymbol{K}x},$$

and from

$$\boldsymbol{N}^{+-}(x+y) = \boldsymbol{N}^{++}(x)\boldsymbol{N}^{+-}(y),$$

by $y \to 0$ we have

$$\boldsymbol{N}^{+-}(x) = \boldsymbol{N}^{++}(x)\boldsymbol{\Psi}.$$

Analysis of the busy period

We still need to find $\boldsymbol{K}$ and $\boldsymbol{\Psi}$.

Let y be the first $+ \to -$ transition in the busy period, then

$$\boldsymbol{\Psi} = \int_{y=0}^{\infty} \underbrace{e^{\boldsymbol{Q}^{++}y}\boldsymbol{Q}^{+-}e^{(\boldsymbol{Q}^{--}+\boldsymbol{Q}^{-+}\boldsymbol{\Psi})y}}_{\boldsymbol{F}(y)} dy$$

where $(\boldsymbol{Q}^{--} + \boldsymbol{Q}^{-+}\boldsymbol{\Psi})$ is the generator of the censored process for the negative states.

For $\boldsymbol{F}(y)$ we have

$$\frac{d}{dy}\boldsymbol{F}(y) = \boldsymbol{Q}^{++}\boldsymbol{F}(y) + \boldsymbol{F}(y)(\boldsymbol{Q}^{--} + \boldsymbol{Q}^{-+}\boldsymbol{\Psi})$$

and

$$\begin{aligned}
&\int_{y=0}^{\infty} \frac{d}{dy}\boldsymbol{F}(y)dy = \boldsymbol{F}(\infty) - \boldsymbol{F}(0) = -\boldsymbol{Q}^{+-} = \\
&= \int_{y=0}^{\infty} \boldsymbol{Q}^{++}\boldsymbol{F}(y)dy + \int_{y=0}^{\infty} \boldsymbol{F}(y)(\boldsymbol{Q}^{--} + \boldsymbol{Q}^{-+}\boldsymbol{\Psi})dy \\
&= \boldsymbol{Q}^{++}\boldsymbol{\Psi} + \boldsymbol{\Psi}(\boldsymbol{Q}^{--} + \boldsymbol{Q}^{-+}\boldsymbol{\Psi})
\end{aligned}$$

which is a Ricatti equation for $\boldsymbol{\Psi}$.

Analysis of the busy period

$p^0_{ij}(x,t) = \dfrac{\partial}{\partial x} P^0_{ij}(x,t)$ satisfies the same PDE as $p_{ij}(x,t)$ for $x > 0$, that is

$\dfrac{\partial}{\partial t} p^0_{++}(x,t) + \dfrac{\partial}{\partial x} p^0_{++}(x,t) = p^0_{++}(x,t)\boldsymbol{Q}^{++} + p^0_{+-}(x,t)\boldsymbol{Q}^{-+}.$

Integrating it from $t = 0$ to ∞ we have

$$\frac{\partial}{\partial x}\boldsymbol{N}_{++}(x) = \boldsymbol{N}_{++}(x)\boldsymbol{Q}^{++} + \boldsymbol{N}_{+-}(x)\boldsymbol{Q}^{-+}.$$

Substituting $\boldsymbol{N}_{++}(x) = e^{\boldsymbol{K}x}$ and $\boldsymbol{N}_{+-}(x) = e^{\boldsymbol{K}x}\boldsymbol{\Psi}$ gives

$$\boldsymbol{K} = \boldsymbol{Q}^{++} + \boldsymbol{\Psi}\boldsymbol{Q}^{-+}.$$

Analysis of the busy period

Additionally, let z be the fluid level at the last $+ \to -$ transition in the busy period, then

$$\boldsymbol{\Psi} = \int_{z=0}^{\infty} \underbrace{e^{\boldsymbol{K}z}\boldsymbol{Q}^{+-}e^{\boldsymbol{Q}^{--}z}}_{\boldsymbol{V}(z)} dz \ .$$

Consequently $\boldsymbol{K}$ and $\boldsymbol{\Psi}$ satisfy

$$-\boldsymbol{Q}^{+-} = \boldsymbol{K}\boldsymbol{\Psi} + \boldsymbol{\Psi}\boldsymbol{Q}^{--}.$$

Process restricted to empty buffer

Restricting the fluid buffer for the time when the buffer is idle we have a CTMC with generator

$$\boldsymbol{Q}^{--} + \boldsymbol{Q}^{-+}\boldsymbol{\Psi}.$$

The stationary distribution of this restricted process is proportional with ℓ^- that is

$$\ell^-(\boldsymbol{Q}^{--} + \boldsymbol{Q}^{-+}\boldsymbol{\Psi}) = 0.$$

The related normalizing condition is

$$\begin{aligned}
1 &= \ell^- \mathbb{1} + \int_x p(x)\mathbb{1}dx = \\
&= \ell^- \mathbb{1} + \int_x \ell^- \boldsymbol{Q}^{-+} e^{\boldsymbol{K}x} [\boldsymbol{I} \quad \boldsymbol{\Psi}] \mathbb{1}dx \\
&= \ell^- \Big(\mathbb{1} + \boldsymbol{Q}^{-+}(-\boldsymbol{K})^{-1} [\boldsymbol{I} \quad \boldsymbol{\Psi}] \mathbb{1} \Big).
\end{aligned}$$

QBD based solution of the Ricatti equation

The Ricatti equation

$$\mathbf{0} = \boldsymbol{Q}^{+-} + \boldsymbol{Q}^{++}\boldsymbol{\Psi} + \boldsymbol{\Psi}\boldsymbol{Q}^{--} + \boldsymbol{\Psi}\boldsymbol{Q}^{-+}\boldsymbol{\Psi}$$

of size $|\mathcal{S}^{+}| \times |\mathcal{S}^{-}|$ can be transformed into a quadratic matrix equation of size $|\mathcal{S}| \times |\mathcal{S}|$

Let $c = \max_{i \in S} |\mathbf{Q}_{ii}|$ and define matrix $\mathbf{P} = \mathbf{I} + \mathbf{Q}/c$ which is identically partitioned as $\mathbf{Q}$. Let

$$\mathbf{F} = \begin{array}{|c|c|}\hline \frac{1}{2}\mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \\ \hline \end{array}, \ \mathbf{L} = \begin{array}{|c|c|}\hline \frac{1}{2}\mathbf{P}_{++} - \mathbf{I} & \mathbf{0} \\ \hline \mathbf{P}_{-+} & -\mathbf{I} \\ \hline \end{array}, \ \mathbf{B} = \begin{array}{|c|c|}\hline \mathbf{0} & \frac{1}{2}\mathbf{P}_{+-} \\ \hline \mathbf{0} & \mathbf{P}_{--} \\ \hline \end{array}.$$

$\boldsymbol{\Psi} = \mathbf{G}_{+-}$ is obtained from the minimal non-negative solution of $\mathbf{B} + \mathbf{L}\mathbf{G} + \mathbf{F}\mathbf{G}^2 = \mathbf{0}$.

Notations: Queues

m	number of servers
$B(t)$	service time distribution
S, T_B	service time r.v.
$A(t)$	inter-arrival time distribution
T_A	inter-arrival time r.v.
W	waiting time r.v.
T	system time r.v. ($T = S + W$)
Q	queue length r.v.
K	number of customers (queue+servers) r.v.
$\overline{X}$	mean of r.v. X
ρ	utilization
c_X^2	squared coefficient of variation of r.v. X

Notations: Markov chains

$\mathbf{Q}$	generator matrix of a CTMC
π	stationary probability vector of a CTMC
$\mathbf{P}$	state transition probability matrix of a DTMC
ν	stationary probability vector of a DTMC

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◦ M. Neuts: Matrix-geometric solutions in stochastic models. An algorithmic approach. The Johns Hopkins University Press, Baltimore, MD, 1981.

Both:

- L. Lakatos, L. Szeidl, and M. Telek, Introduction to Queueing Systems with Telecommunication Applications. Springer, 2013.