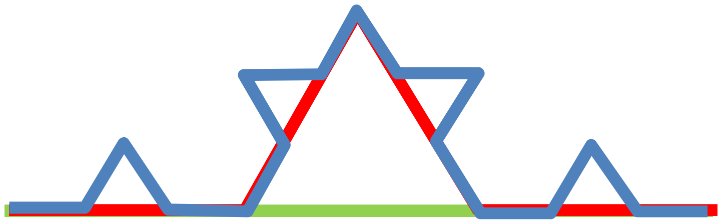
Fraktálok

Szirmay-Kalos László

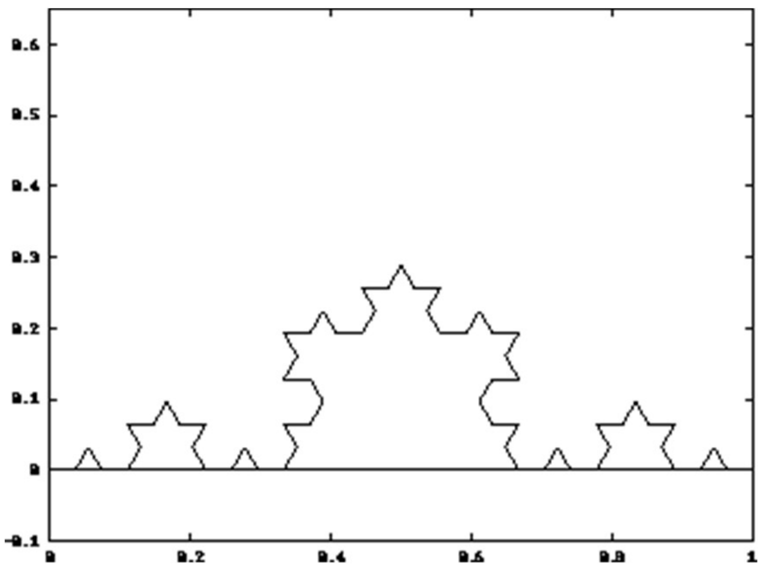
A természet geometriája

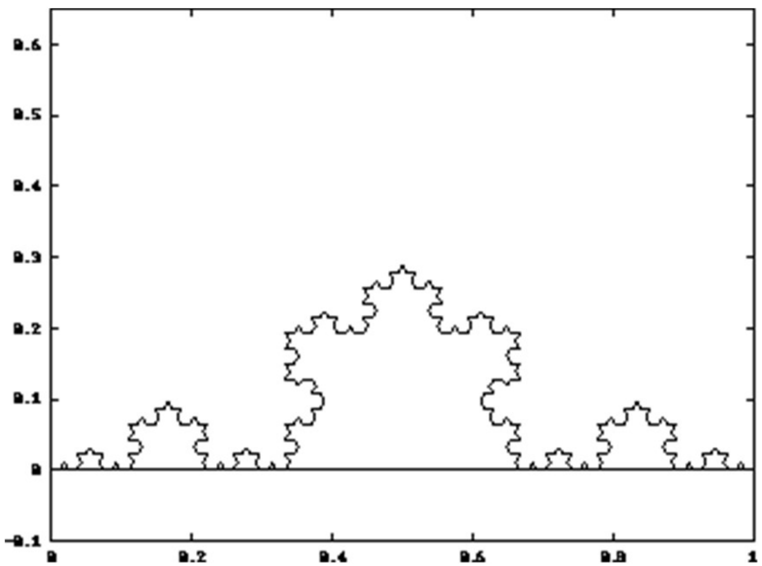
- Euklideszi geometria metrikus
  - „Sima” egyenesre épít (analízis: differenciálás)
  - Méret lényeges (milyen méret: dimenzió kapcsolat)
  - Hossz (1D), felszín (2D), térfogat (3D)
  - Ember alkotta objektumokra jó
- Természet
  - Méret nem releváns (zérus vagy végtelen):
    - dimenzió?
  - Nem sima (nem differenciálható)

Koch görbe: hossz végtelen



$$l_n = l_0 \left(\frac{4}{3}\right)^n \to \infty$$



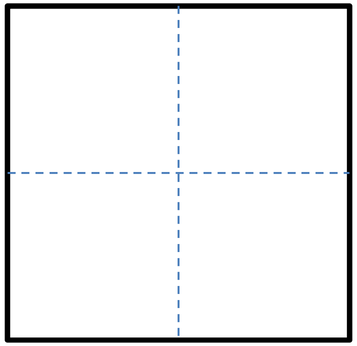


Hausdorff dimenzió önhasonló objektumokra

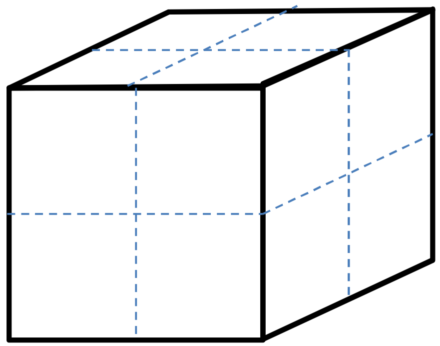
$r = 1/2$

$N = 2$

$N = 4$



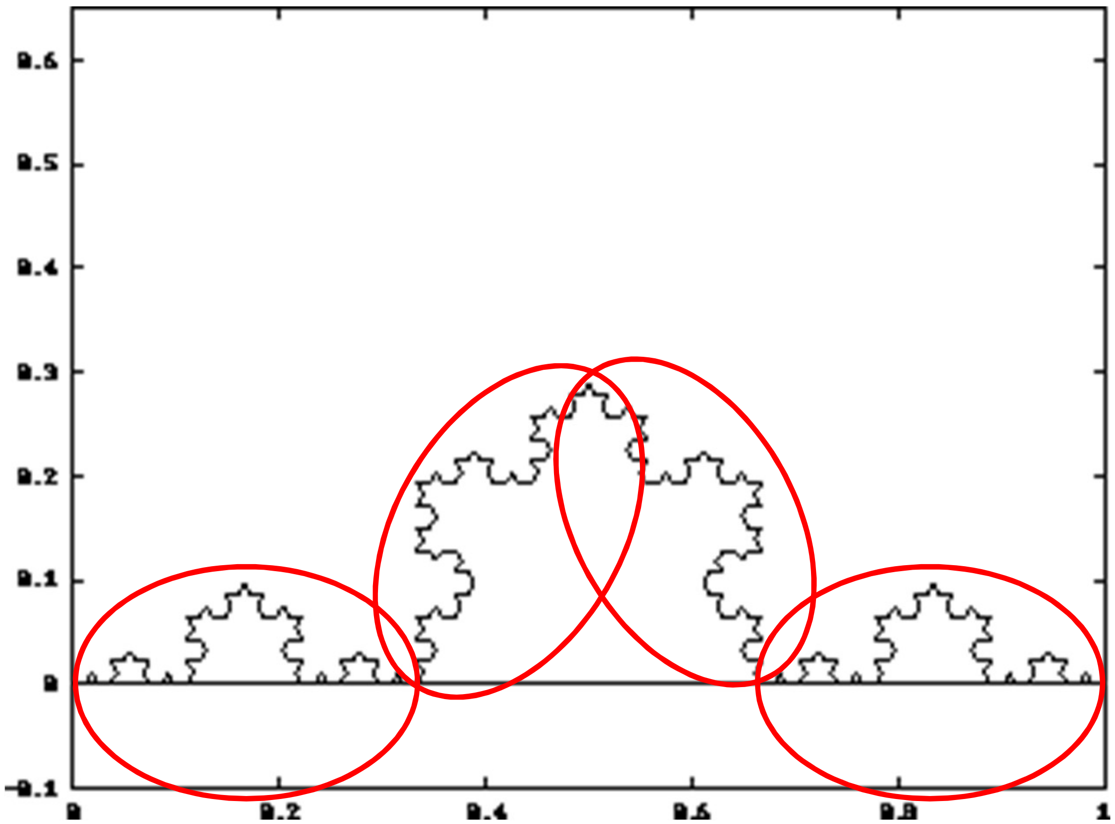
$N = 8$



$N = 1/r^D$

$$D = \frac{\log(N)}{\log(1/r)}$$

Koch görbe Hausdorff dimenziója

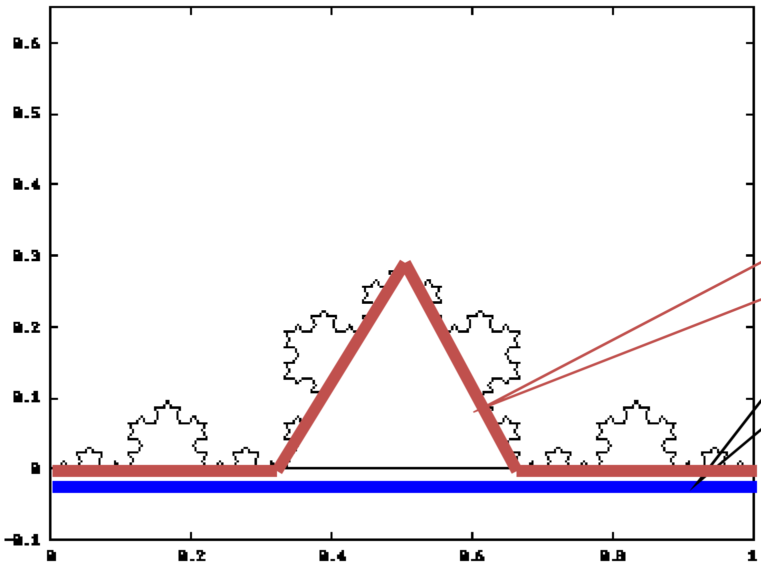


$$r = 1/3$$
$$N = 4$$

$$D = \frac{\log(4)}{\log(3)} \approx 1.26$$

Nem önhasonló objektumok: vonalzó dimenzió

$$N = 1/r^D$$



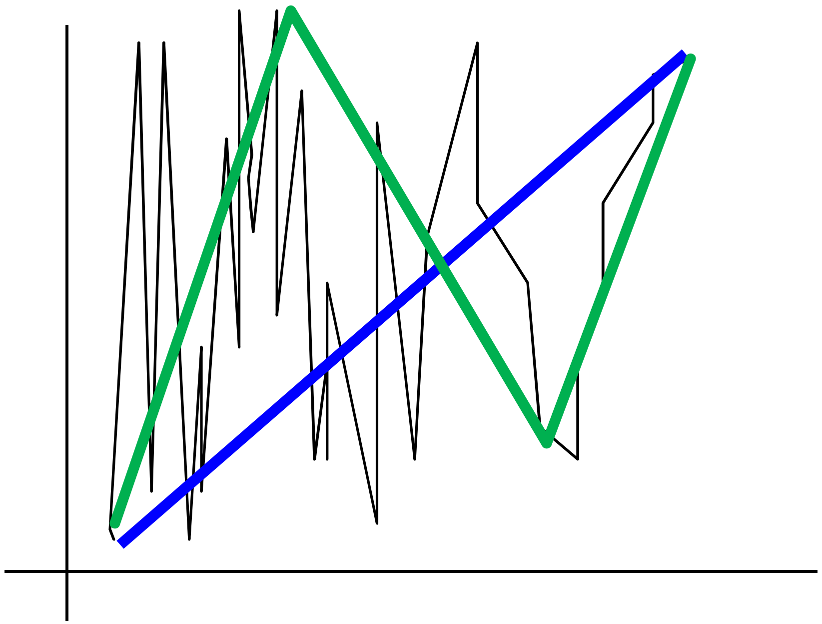
Vonalzó (l)	db
1	1
$r = 1/3$	$N = 4$
r^2	N^2
r^m	N^m

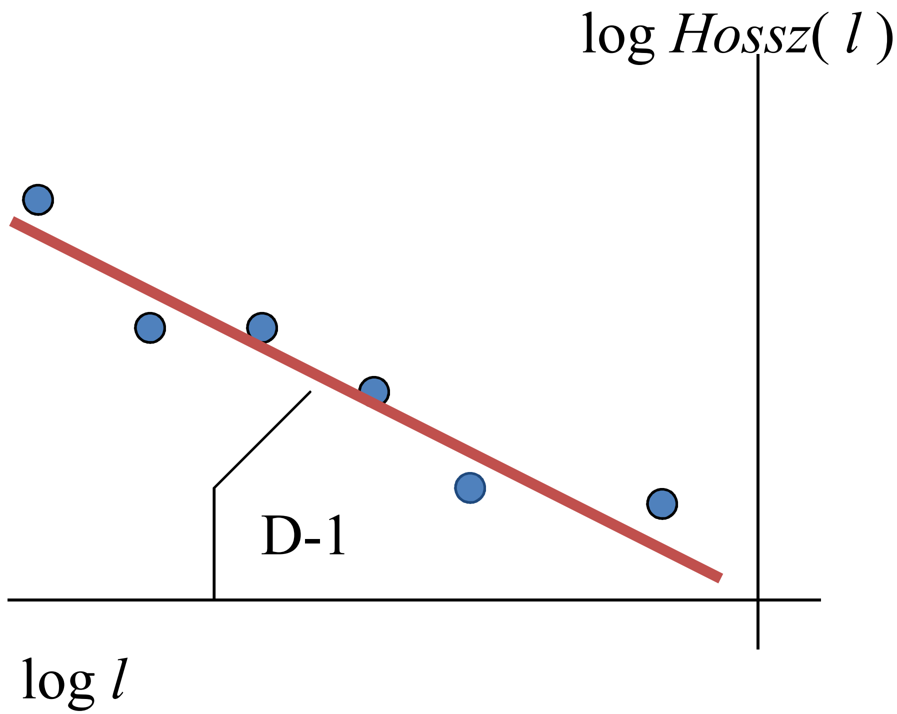
$$Hossz(l) = l \text{ db} = l\, N^m = l\,(1/r^D)^m =$$
$$= l\,(1/r^m)^D = 1/l^{D-1}$$

$$\boxed{D = -\log(Hossz(l))/\log(l) + 1}$$

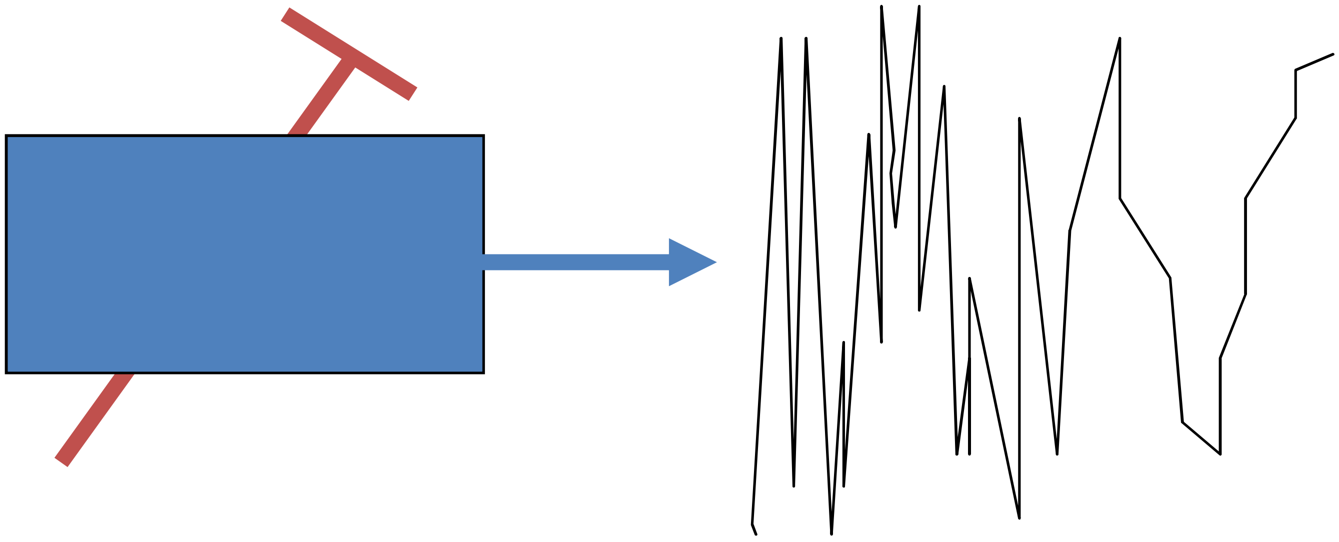
Dimenziómérés = hosszmérés

Pl. EEG görbe,
Anglia v. Horvátország partja



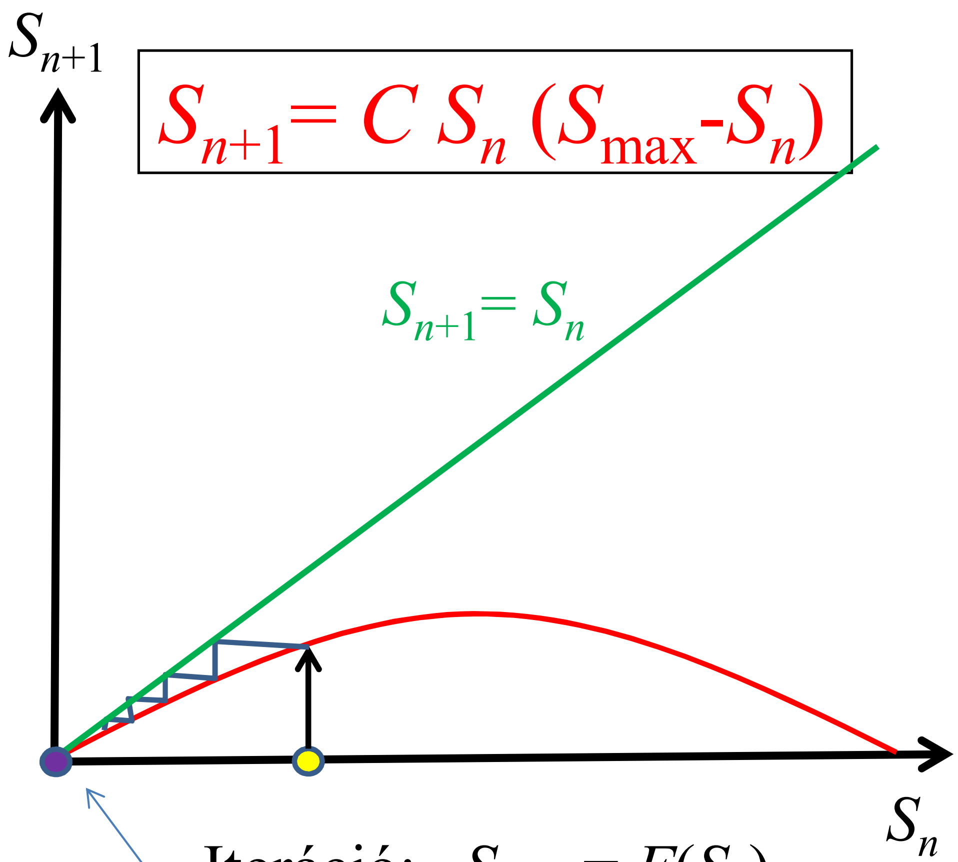


Fraktálok előállítása



Matematikai gépek:
- Brown mozgás
- Kaotikus dinamikus rendszerek
  (véletlenszám generátor)

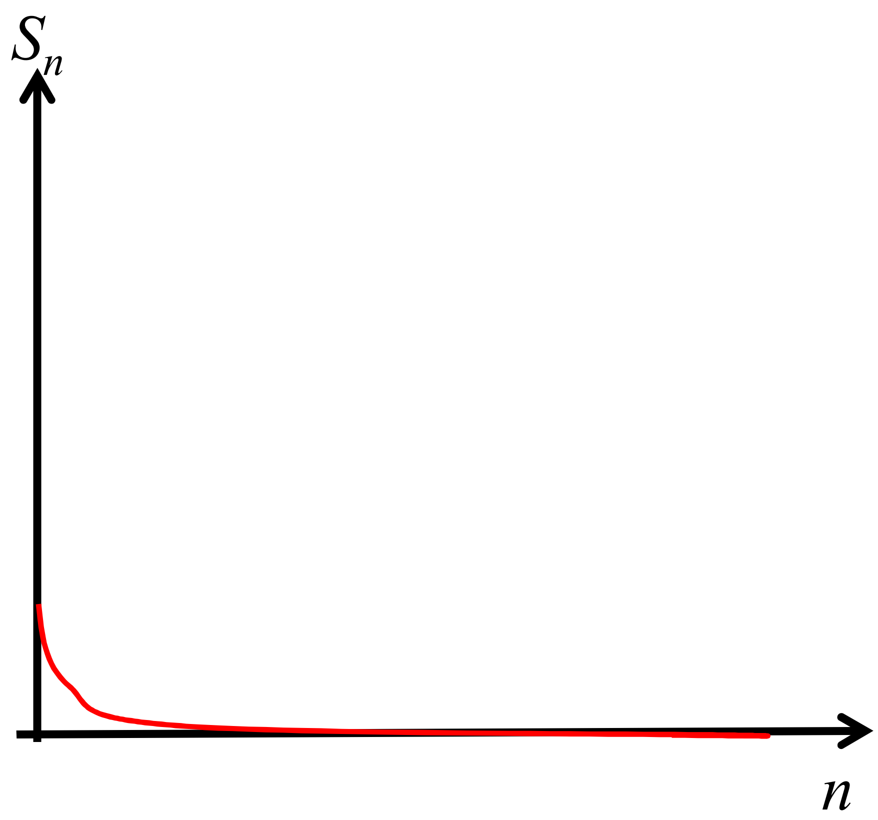
Kaotikus dinamikus rendszer: nyulak kis *C* értékre



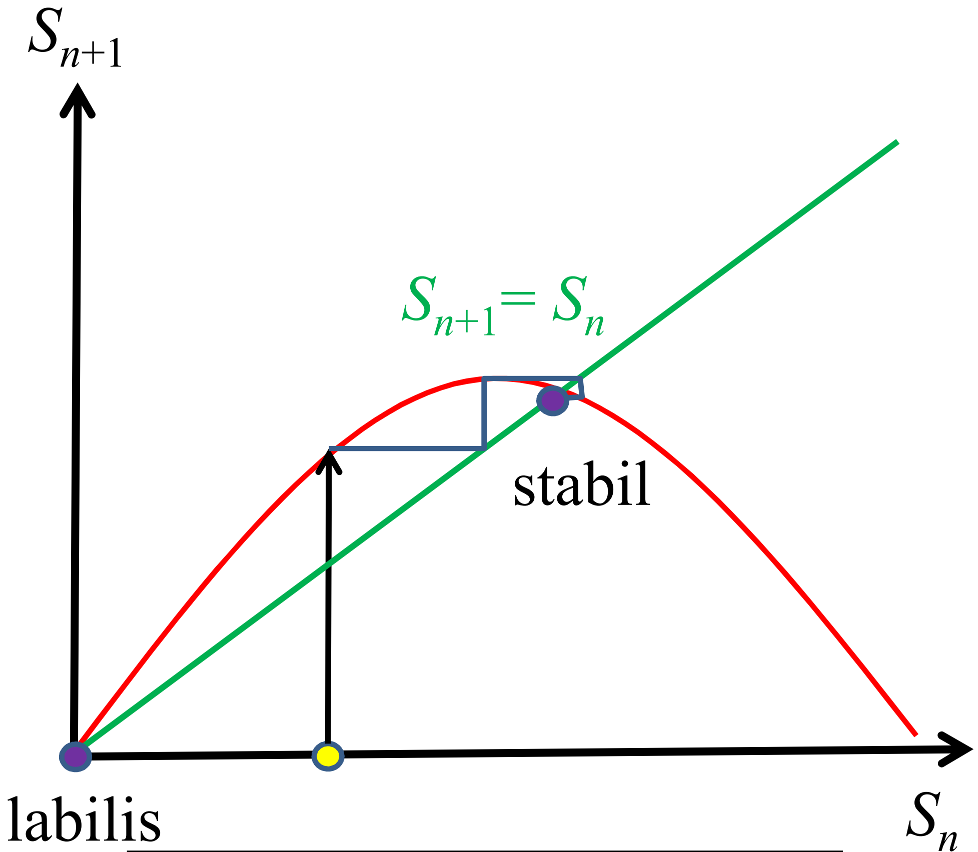
Iteráció: $S_{n+1} = F(S_n)$

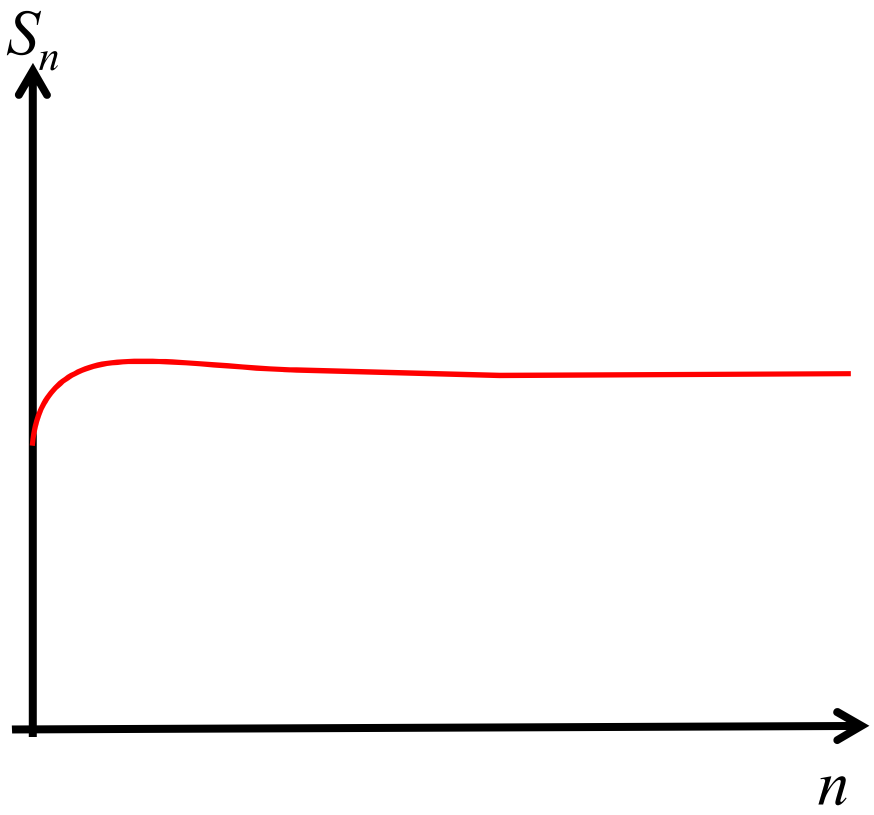
Fix pont: $S^* = F(S^*)$

Stabil: $|F'(S^*)| < 1$



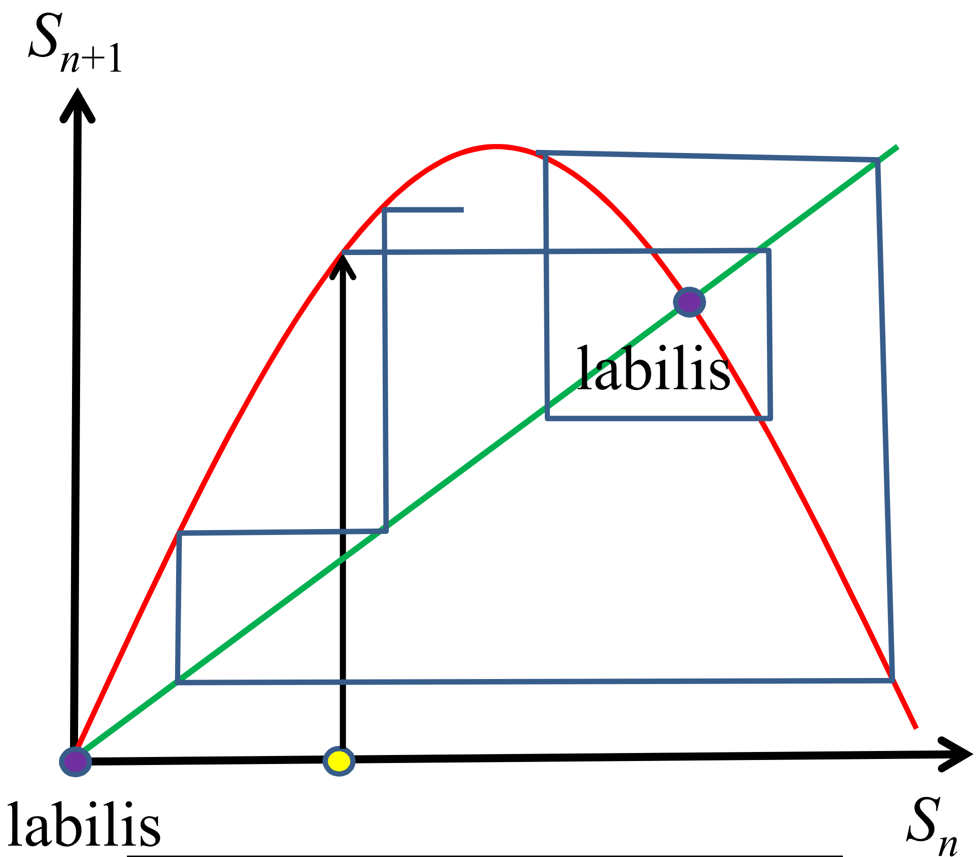
Kaotikus dinamikus rendszer: nyulak közepes *C* értékre



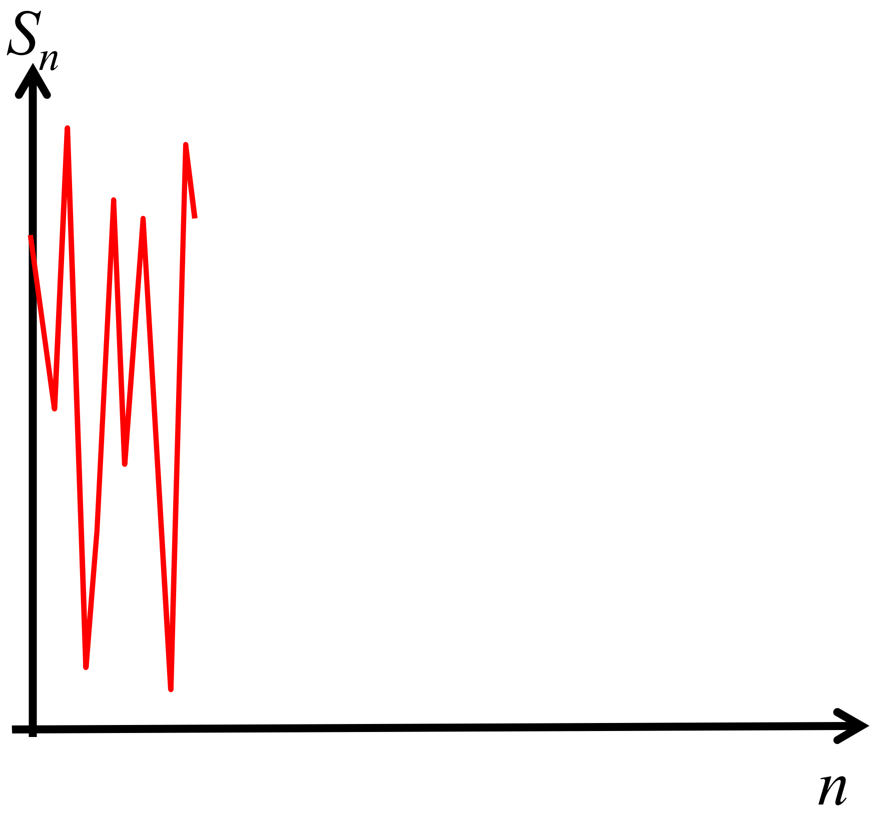


$$\boxed{S_{n+1} = C\, S_n\, (S_{max} - S_n)}$$

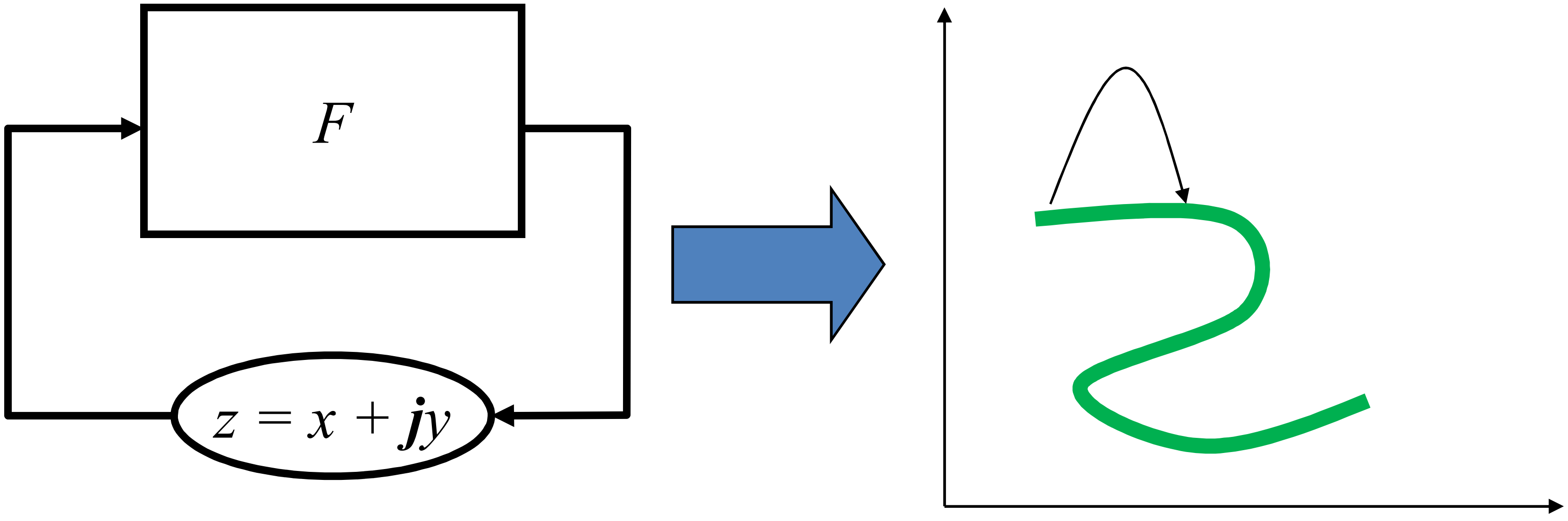
Kaotikus dinamikus rendszer: nyulak nagy *C* értékre



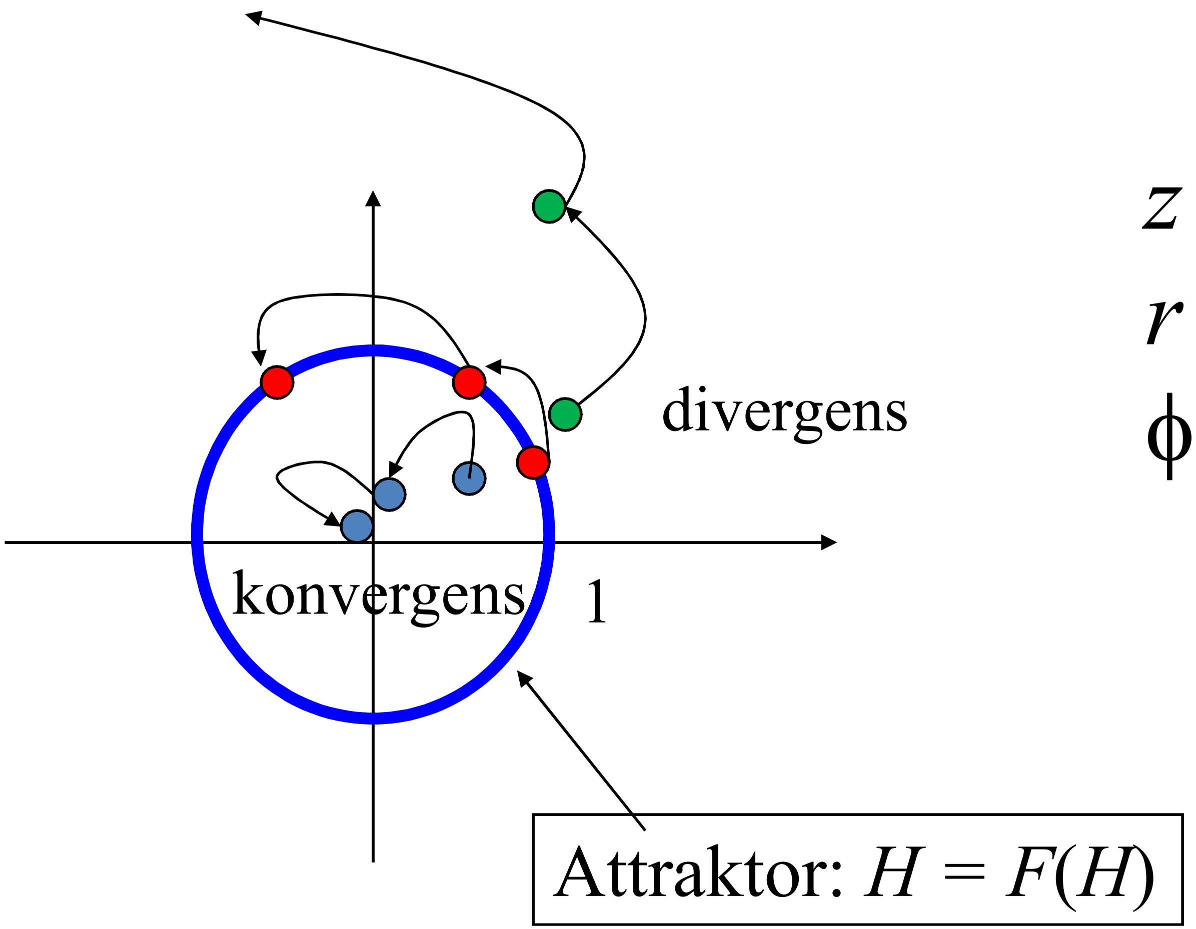
$$\boxed{S_{n+1} = C\, S_n\, (S_{max} - S_n)}$$



Kaotikus rendszerek a síkon



$F: z \to z^2$



$$z = r\,\mathrm{e}^{j\phi}$$

$$r \to r^2$$

$$\phi \to 2\phi$$

Attraktor felrajzolása

- Attraktor a labilis és a stabil tartomány határa: kitöltött attraktor = nem divergens pontok
  - $z_{n+1} = z_n^2$: ha $z_\infty < \infty$ akkor fekete
- Attraktorhoz konvergálunk, ha az stabil
  - $z_{n+1} = z_n^2$ attraktora labilis

Inverz iterációs módszer

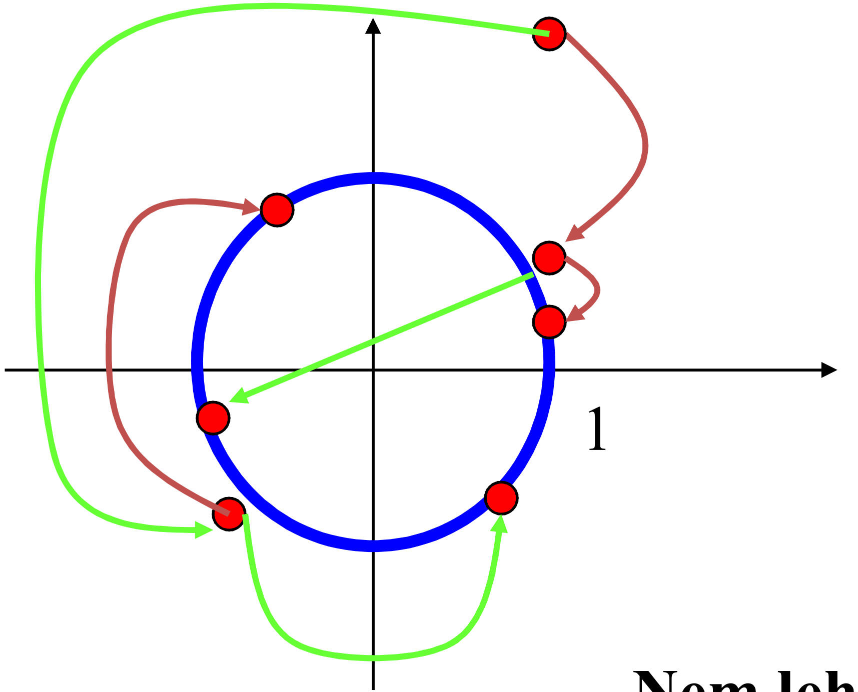
$$H = F(H) \qquad \rightarrow \qquad H = F^{-1}(H)$$

$$z_{n+1} = z_n^{\,2}$$

$$z_{n+1} = \pm\sqrt{z_n}$$

$$r_{n+1} = \sqrt{r_n}$$

$$\phi_{n+1} = \phi_n/2 + \{0,1\}\cdot\pi$$



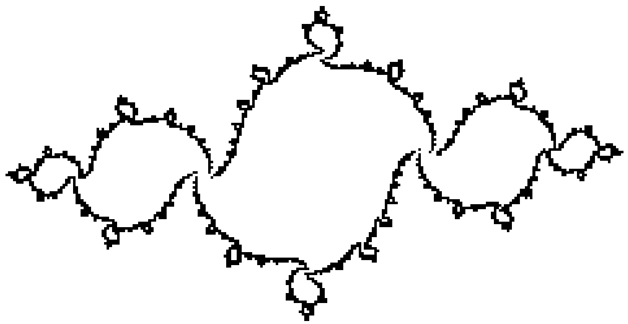
$$r_n \approx 1$$

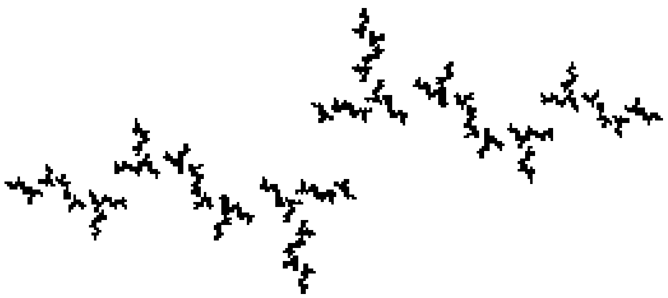
$$\phi_n \approx \underset{n}{\{0,1\}}.\underset{n-1}{\{0,1\}}\underset{n-2}{\{0,1\}}\ldots\cdot\pi$$

Nem lehet csak egy értékkel dolgozni ???

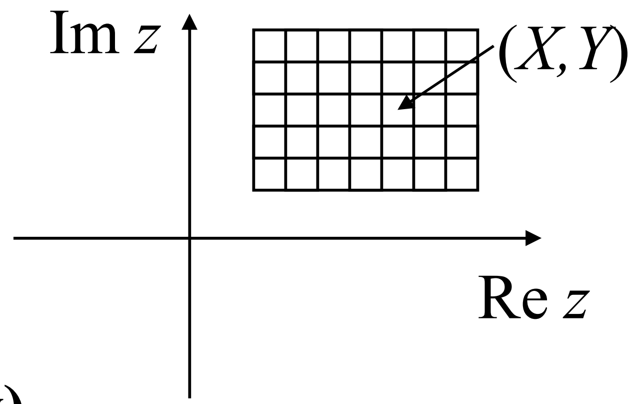
Julia halmaz

$$F: z \rightarrow z^2 + c$$



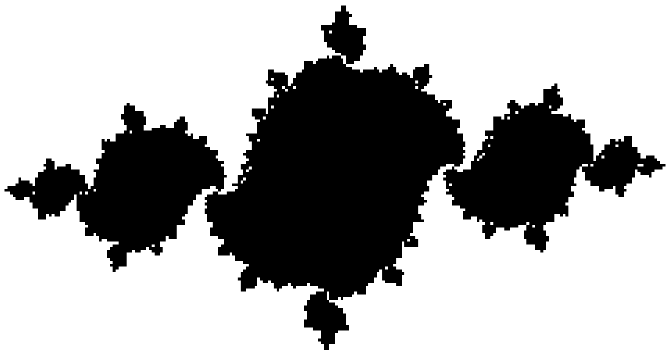


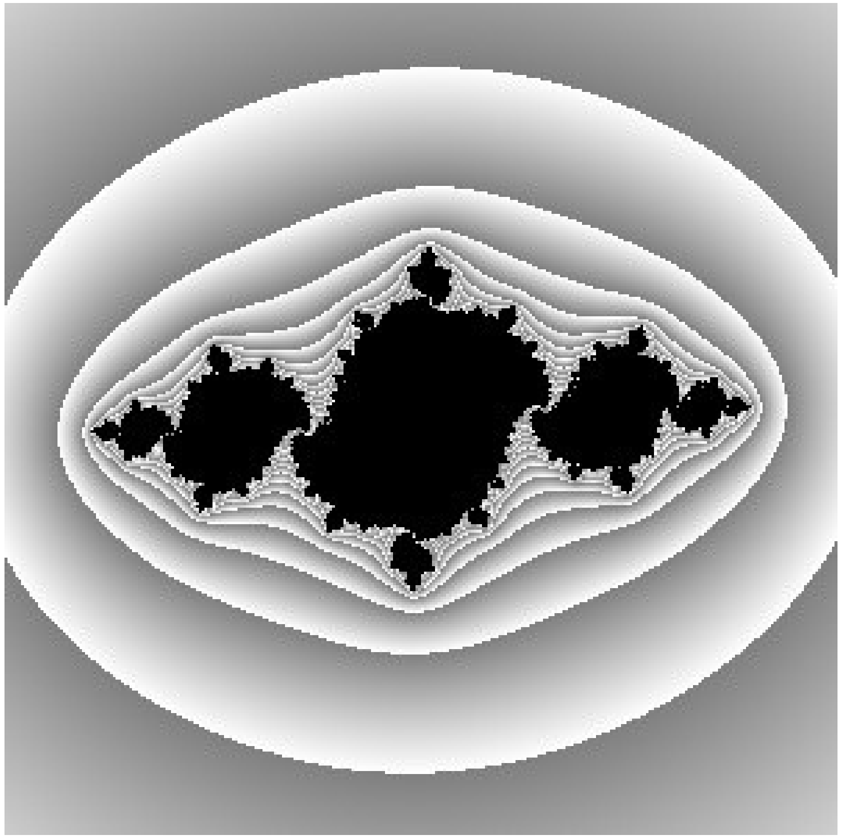
Kitöltött Julia halmaz: algoritmus



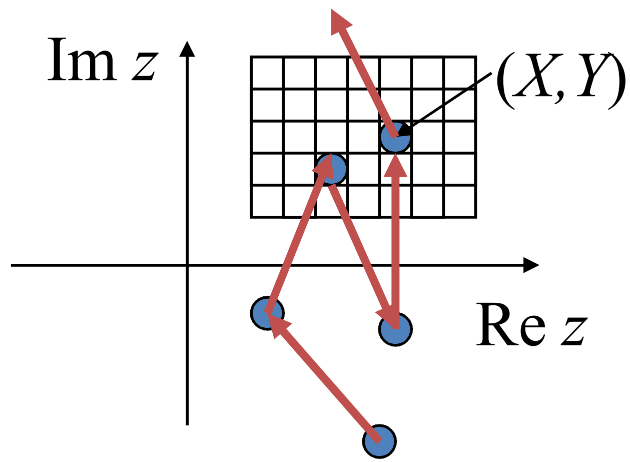

```
FilledJuliaDraw ( )
    FOR Y = 0 TO Ymax DO
        FOR X = 0 TO Xmax DO
            ViewportWindow(X,Y→ x, y)
            z = x + j y
            FOR i = 0 TO n DO z = z² + c
            IF |z| > "infinity" THEN  WRITE(X,Y, white)
            ELSE                      WRITE(X,Y, black)
        ENDFOR
    ENDFOR
END
```

Kitöltött Julia halmaz: kép





Julia halmaz inverz iterációval

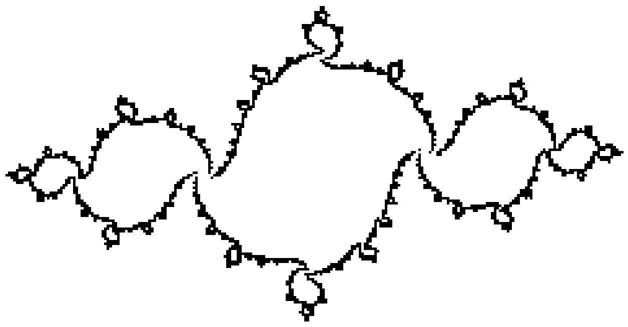


<u>JuliaDrawInverseIterate</u> ( )

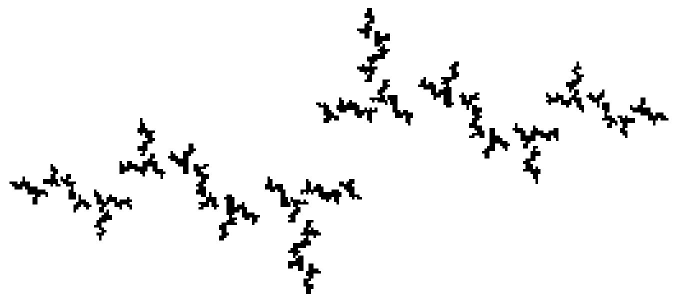
```
JuliaDrawInverseIterate ( )
        Kezdeti z érték választás
        FOR i = 0 TO n DO
                x = Re z,   y = Im z
                IF ClipWindow(x, y)
                        WindowViewport(x, y ⇨ X, Y)
                        Pixel(X, Y) = fekete
                ENDIF
                z = √(z - c)
                if (rand( ) > 0.5) z = -z
        ENDFOR
END
```

Kezdeti z érték:
$z^2 = z - c$ gyöke

Julia halmaz összefüggése

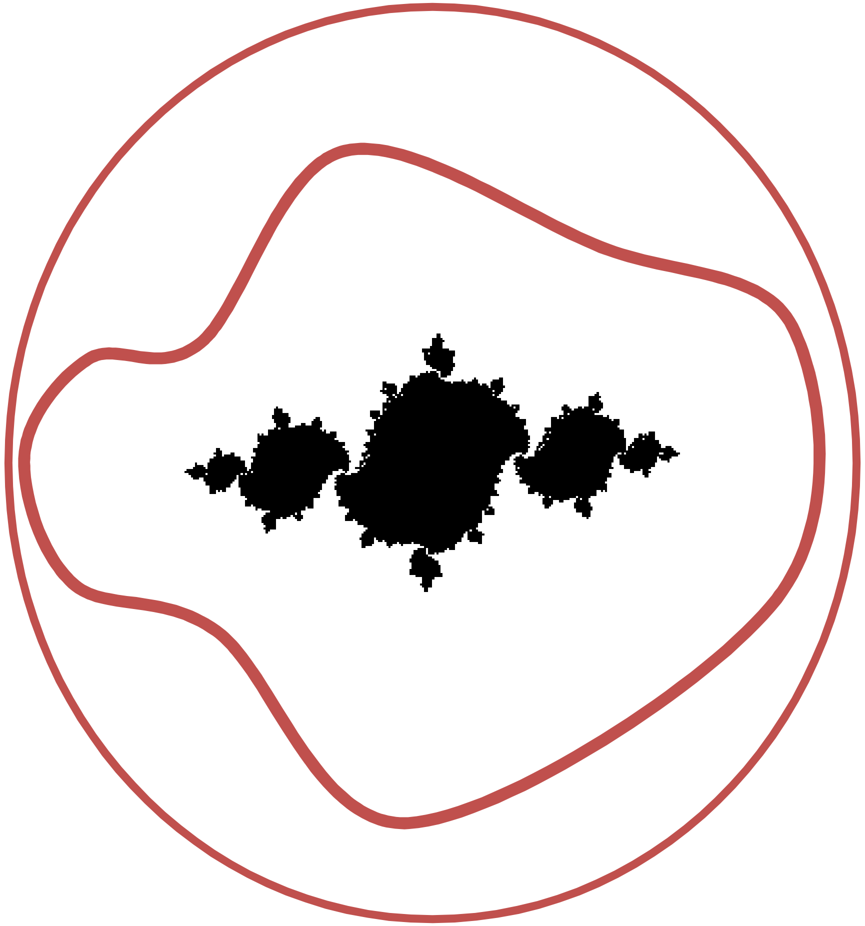


összefüggő

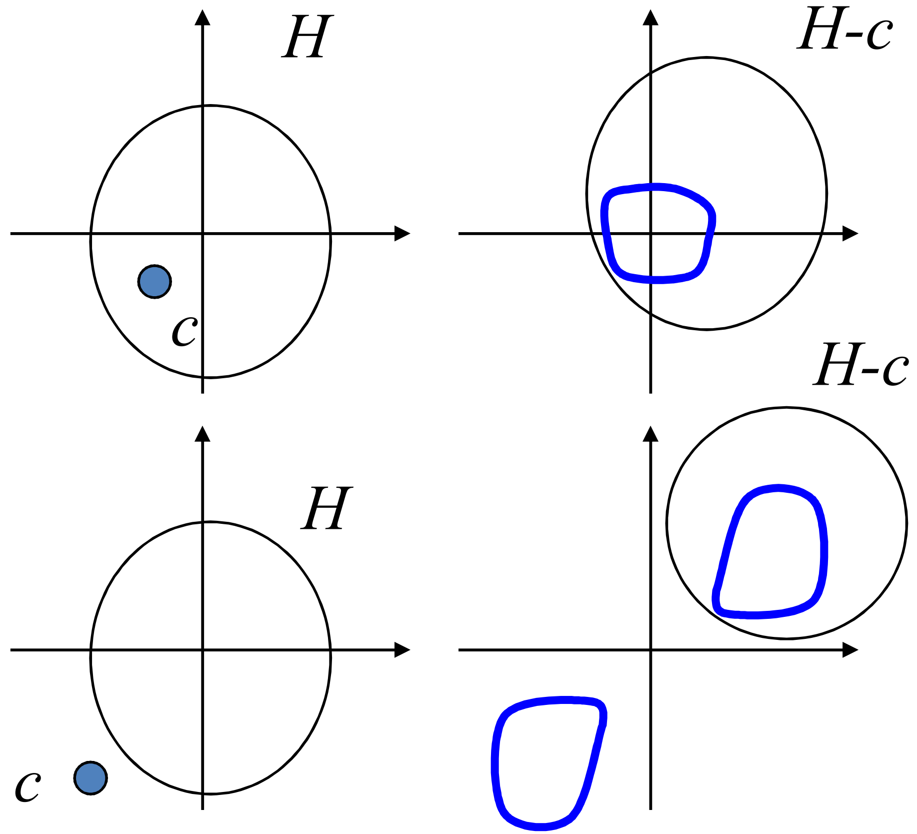


nem összefüggő,
Cantor féle halmaz

Julia halmaz összefüggősége

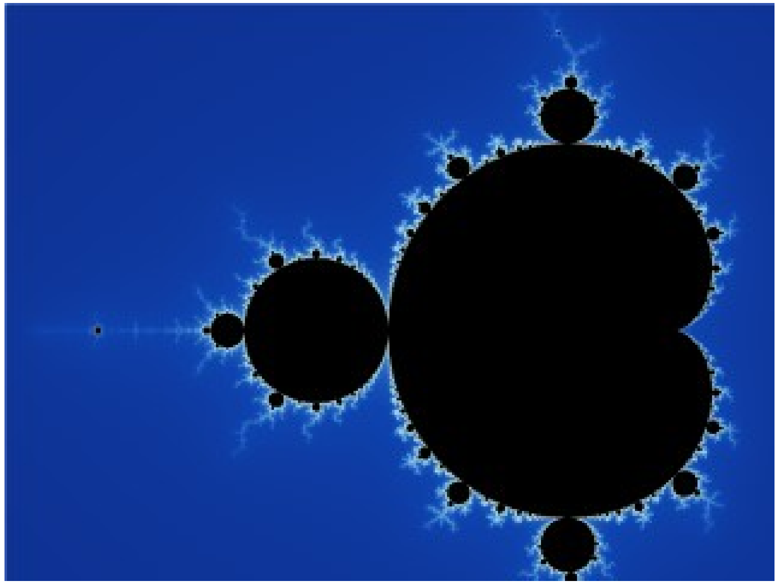


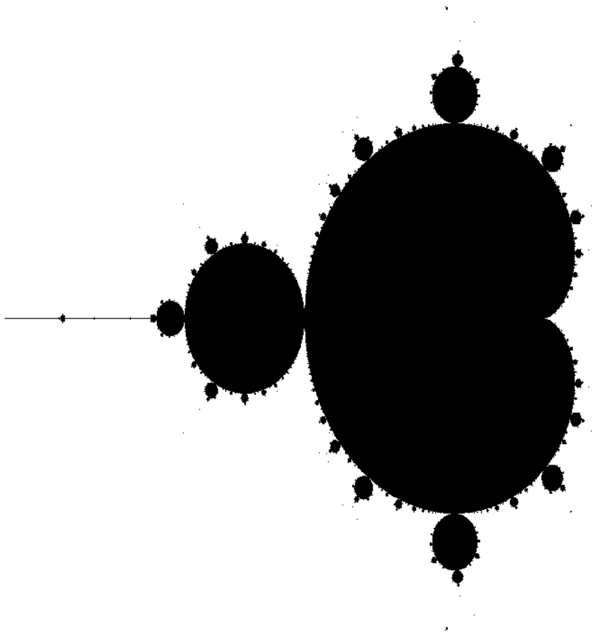
$$z_{n+1} = \pm\sqrt{z_n - c}$$

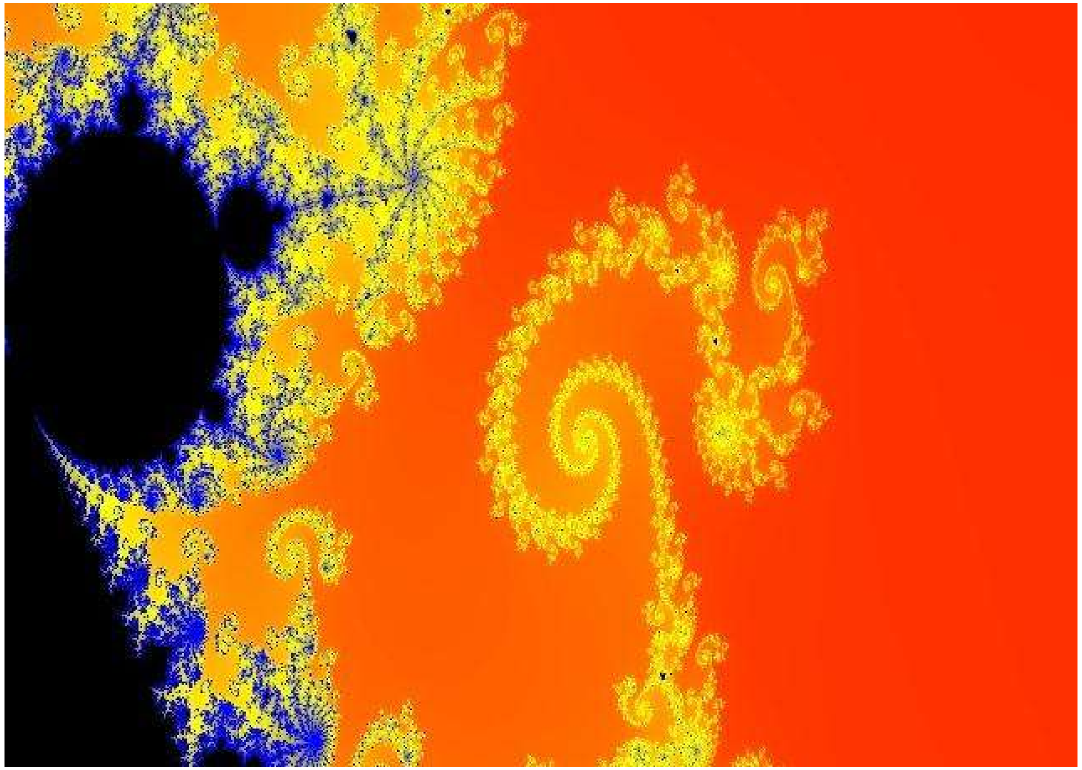


Mandelbrot halmaz

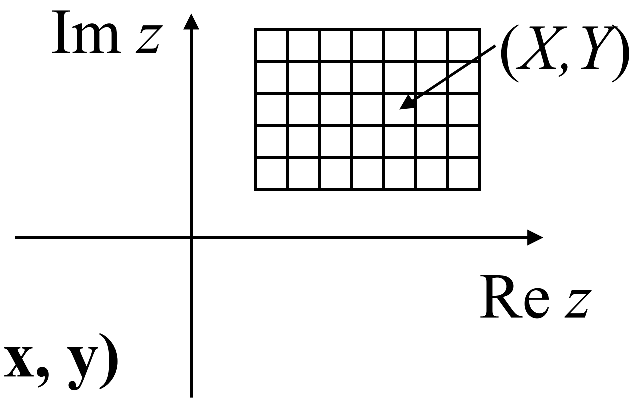
Azon c komplex számok, amelyekre a $z \to z^2 + c$ Julia halmaza összefüggő





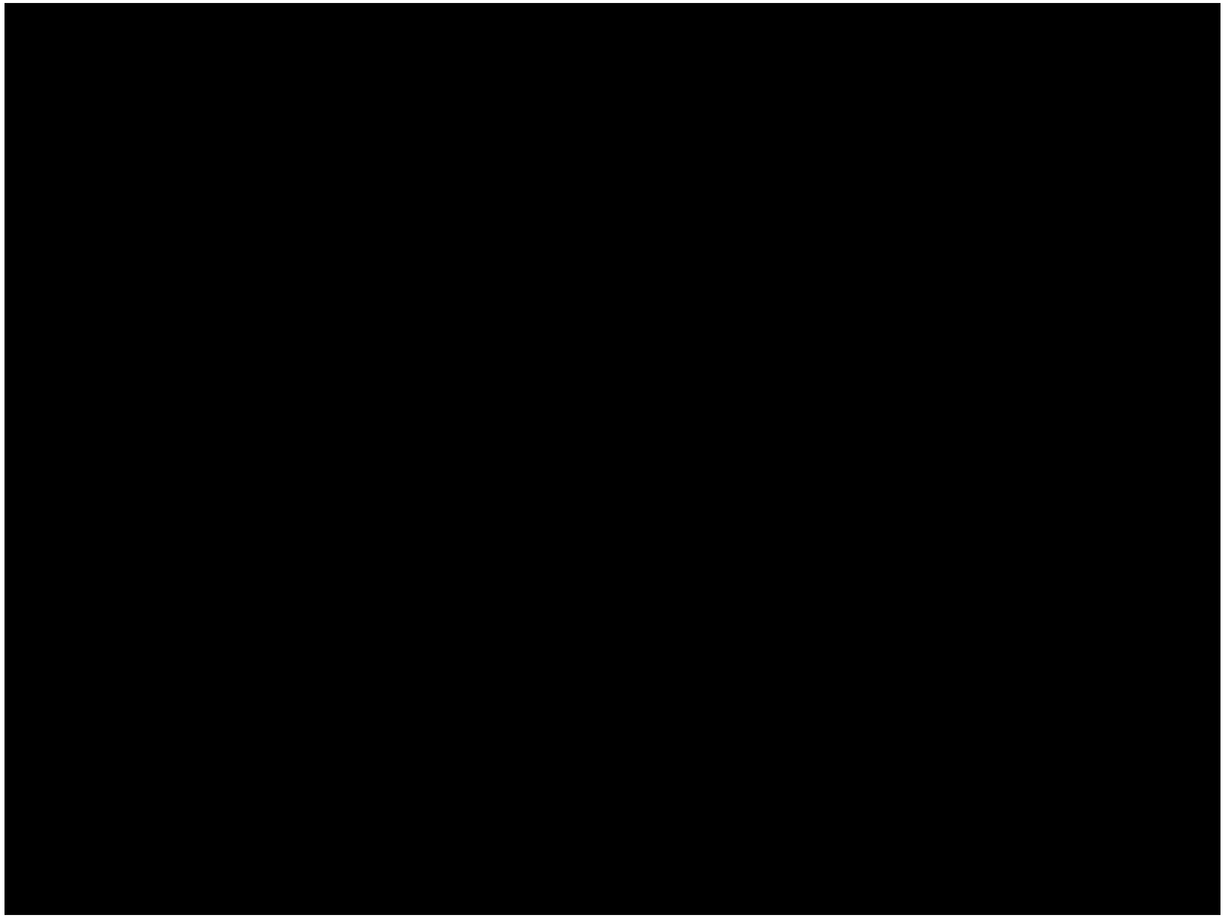


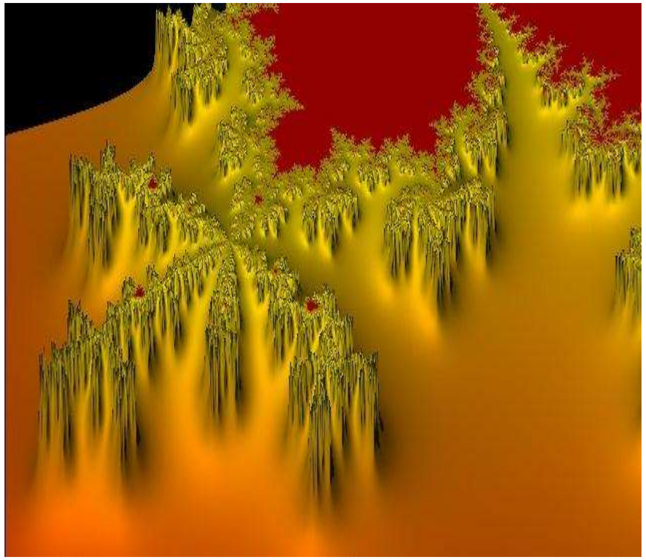
Mandelbrot halmaz, algoritmus

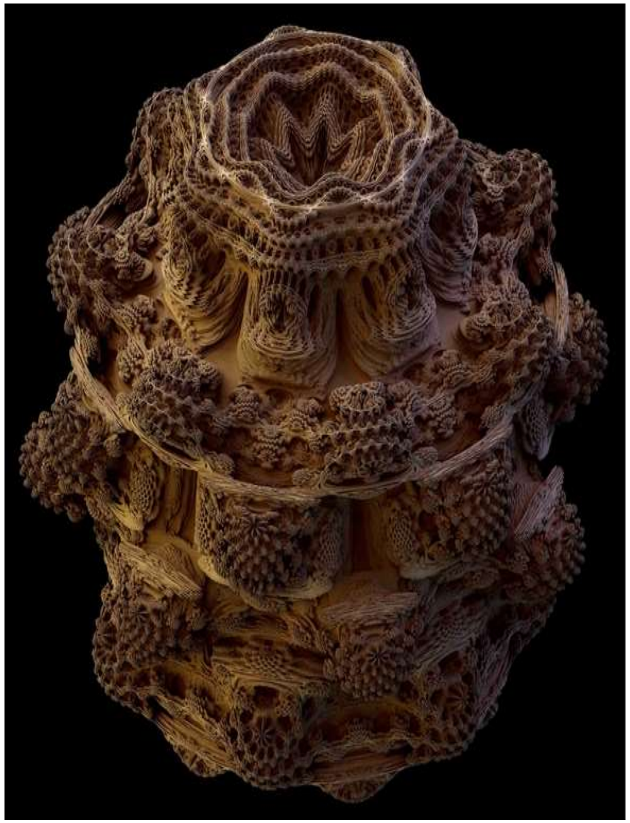



```
MandelbrotDraw ( )
    FOR Y = 0 TO Ymax DO
        FOR X = 0 TO Xmax DO
            ViewportWindow(X,Y→ x, y)
            c = x + j y
            z = 0
            FOR i = 0 TO n DO z = z² + c
            IF |z| > "infinity" THEN  WRITE(X,Y, white)
            ELSE                      WRITE(X,Y, black)
        ENDFOR
    ENDFOR
END
```

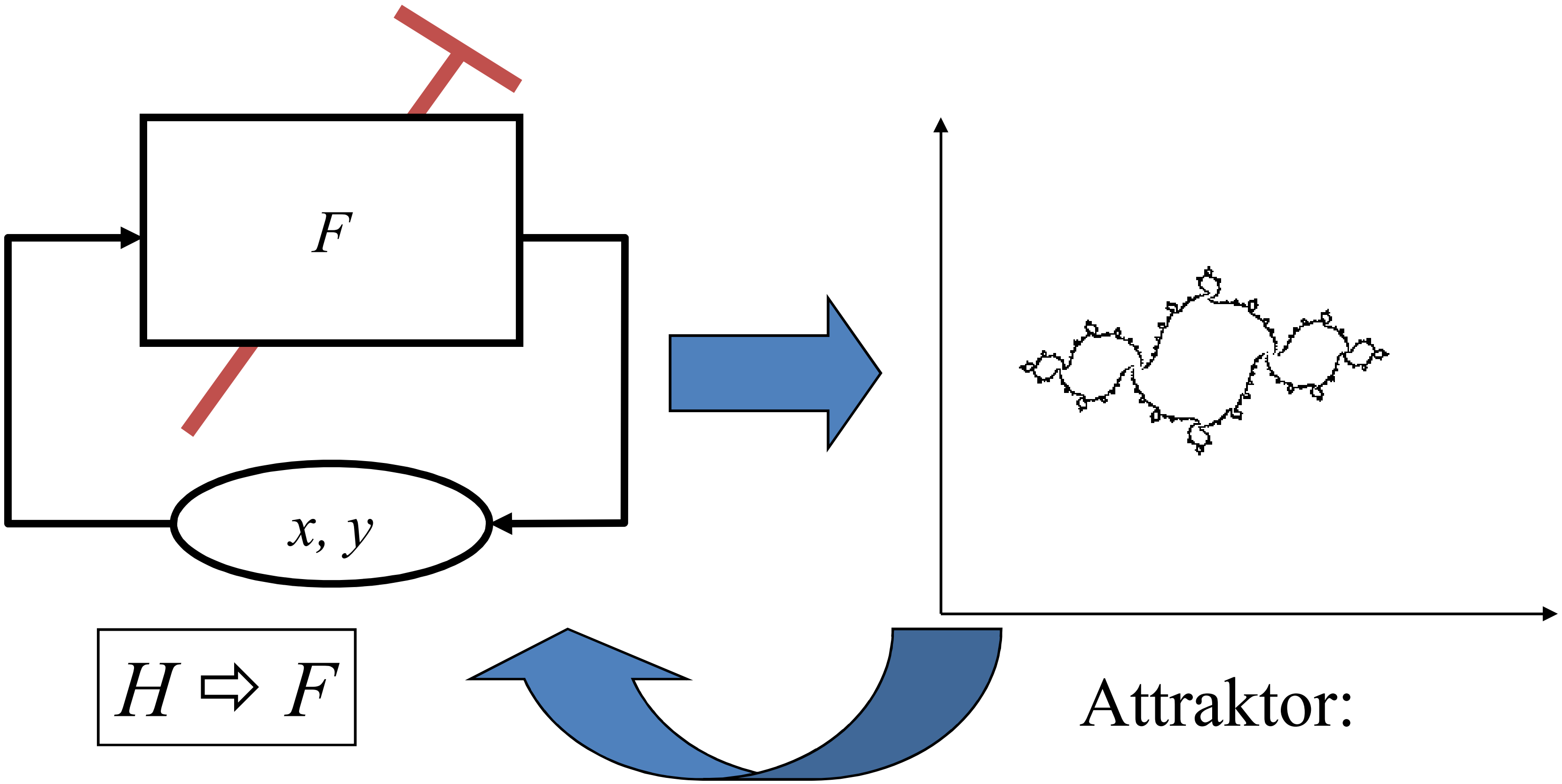
„Matematikát kijátszó”
Mandelbrot halmazok







Inverz feladat: IFS modellezés



Attraktor:

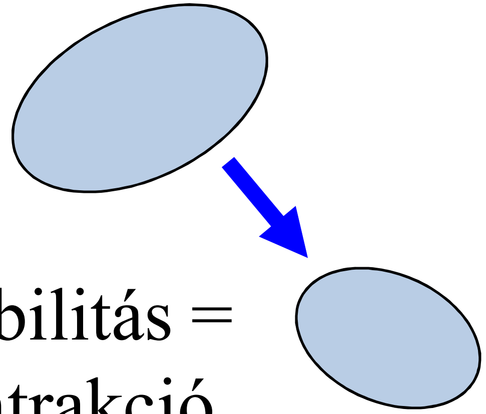
$$H = F(H)$$

F: szabadon vezérelhető, legyen stabil attraktora

F: többértékű lineáris leképzés

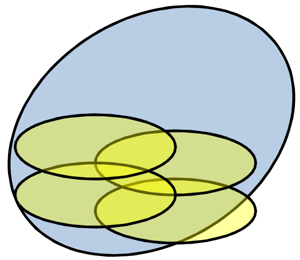
$$F = W_1 \vee W_2 \vee \ldots \vee W_m$$

$$W(x,y) = [ax + by + c,\ dx + ey + f]$$



Stabilitás =
kontrakció

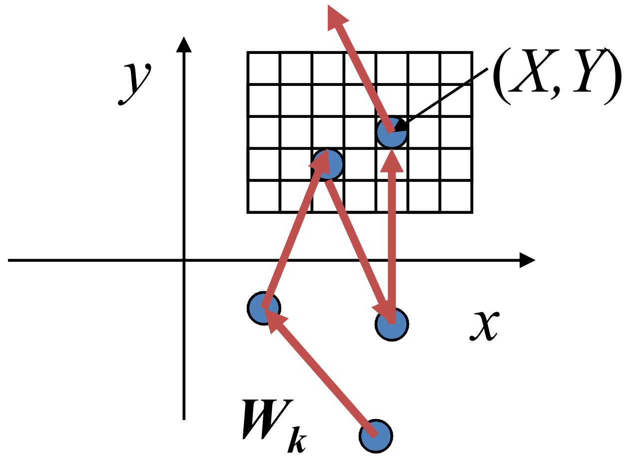
$$\boxed{H = W_1(H) \cup W_2(H) \cup \ldots \cup W_m(H)}$$



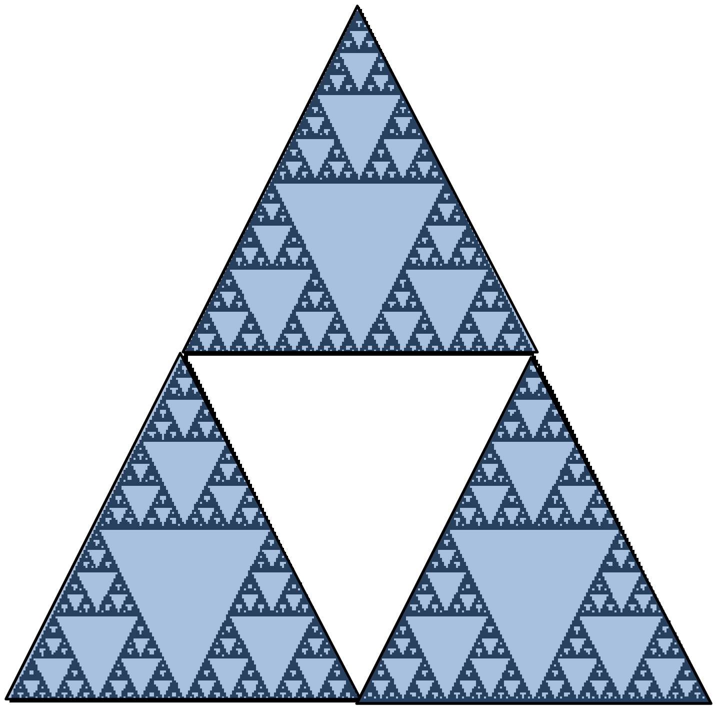
$$H = F(H)$$

IFS rajzolás: iterációs algoritmus

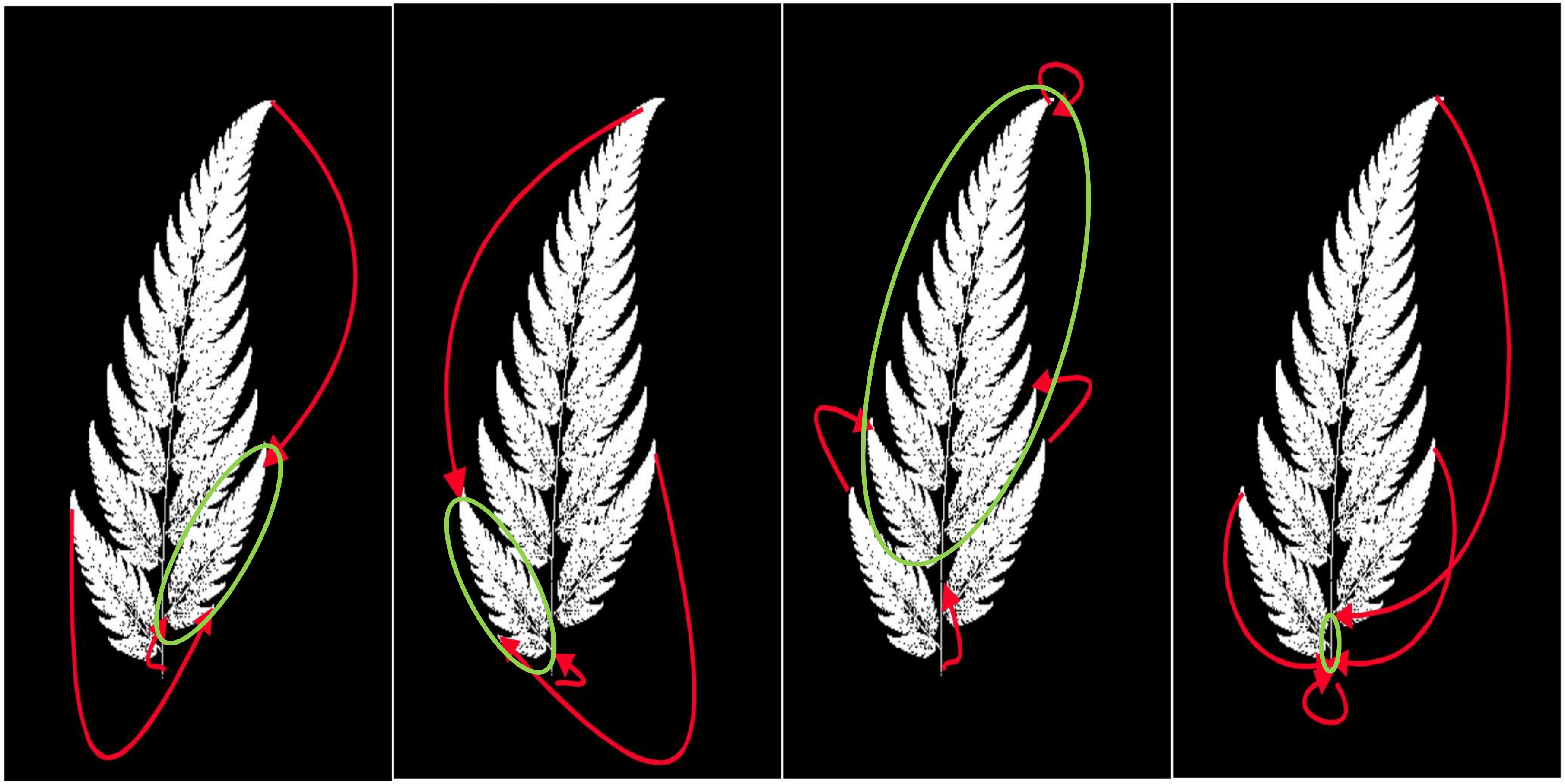
```
IFSDraw ( )
    Legyen [x,y] = [x,y] A₁ + q₁ megoldása a kezdő [x,y]
    FOR i = 0 TO n DO
        IF ClipWindow(x, y)
            WindowViewport(x, y → X, Y)
            Write(X, Y, color);
        ENDIF
        Válassz k-t pₖ valószínűséggel
        [x,y] = [x,y] Aₖ + qₖ
    ENDFOR
END
```



Egyszerű IFS-ek



IFS modellezés



IFS képek